Topological Field Theories
and
Graphical Calculus

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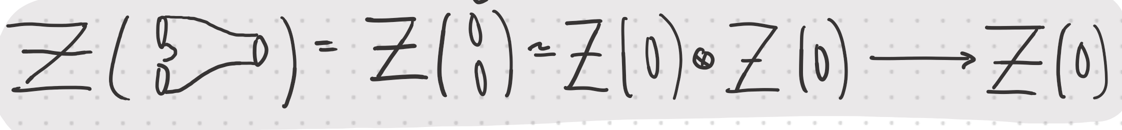
Why?

Topological Field Theories model scale-invariant physical systems:

$$\text{space(time)} \quad M \longmapsto Z(M) \quad \text{states / operators}$$

They're a compact way to describe algebraic structures



Especially gluing properties:

Graphical Calculus describes certain categories with certain embedded graphs

Which Graphs?
Embedded Where?

$\longleftrightarrow$

Structures + Properties
of the
Category

Ribbon Categories and Ribbon Graphs

A ribbon category $\mathcal{A}$ is:

- **Monoidal** $\otimes : \mathcal{A}\times\mathcal{A} \longrightarrow \mathcal{A}$, $\mathbb{1}\otimes - \simeq - \otimes\mathbb{1} \simeq id$
- **braided** $c_{V,W} : V\otimes W \xrightarrow{\simeq} W\otimes V$ such that

$$\begin{tikzcd}
 & W\otimes V\otimes U \arrow[r, "id\otimes c"] & W\otimes U\otimes V \arrow[dr, "c\otimes id"] & \\
V\otimes W\otimes U \arrow[ur, "c\otimes id"] \arrow[dr, "id\otimes c"'] & & & U\otimes W\otimes V \\
 & V\otimes U\otimes W \arrow[r, "c\otimes id"] & U\otimes V\otimes W \arrow[ur, "id\otimes c"'] &
\end{tikzcd}$$

- **rigid**: ${}^*V\otimes V \xrightarrow{b} \mathbb{1} \xrightarrow{d} V\otimes {}^*V$ $\quad V\otimes V^* \xrightarrow{b} \mathbb{1} \xrightarrow{d} V^*\otimes V$

$$\begin{tikzcd}
 & \mathbb{1}\otimes V \simeq V \\
V\otimes V^*\otimes V \arrow[ur, "b\otimes id"] & \\
 & V\otimes\mathbb{1} \simeq V \arrow[ul, "id\otimes d"] \arrow[uu, "id"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & V\otimes\mathbb{1} \simeq V \\
V\otimes V^*\otimes V \arrow[ur, "id\otimes b"] & \\
 & \mathbb{1}\otimes V \simeq V \arrow[ul, "d\otimes id"] \arrow[uu, "id"']
\end{tikzcd}$$

+ equiv. for ${}^*V$

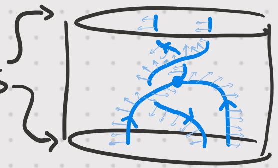
+ compatibility conditions.

- **Pivotal**: $V^{**}\simeq V \iff V^* \xrightarrow[\theta_V]{\simeq} {}^*V$

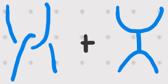
Example: Rib

· up to isotopy!

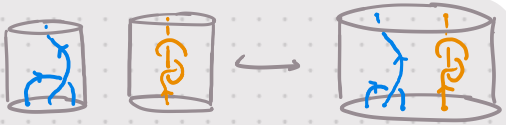
Morphisms: framed oriented graphs in $\mathbb{D}\times[0,1]$

objects 

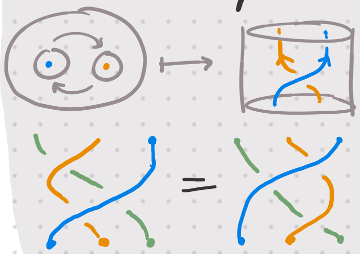
– we'll take formal sums:



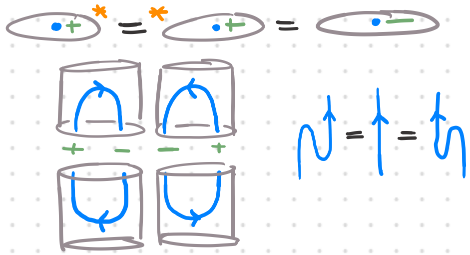
Product induced by embedding:



Braiding induced by isotopy:



Orientation describes duals:



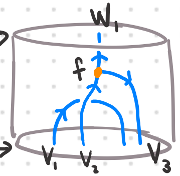
Framing describes pivotal morphism:

Modelling Other Ribbon Categories

Let $\mathcal{A}$ be a ribbon category

Def $\mathrm{Rib}_{\mathcal{A}}$ the cat. of $\mathcal{A}$-coloured ribbon graphs

"$\mathcal{A}$ labellings"



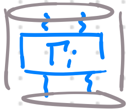
at vertices:

$f : \bigotimes V_i \to \bigotimes V_i$
incoming outgoing

Theorem [Reshetikhin–Turaev '90] There's a unique ribbon functor $\mathrm{Rib}_{\mathcal{A}} \xrightarrow{ev} \mathcal{A}$ such that:

$ev($ V $) = V$ $\qquad ev($ f $) = f$

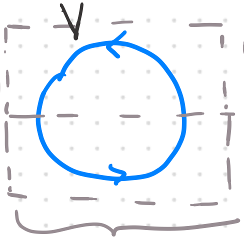
Def (Skein Relations)



$\sum_{i=1}^{n}$ Γ_i $\sim 0 \iff ev\left(\sum_{i=1}^{n} \Gamma_i\right) = 0$

Fact: $\mathrm{Rib}_{\mathcal{A}}/\sim \;\cong \mathcal{A}$

Example: Rep G



$V^* \otimes V \to \mathbb{C}$
$f \otimes v \mapsto f(v)$

so $\overset{V}{\bigcirc} \sim \dim V$

$\mathbb{C} \to V^* \otimes V$
$1 \mapsto \sum_{i=1}^{n} e^i \otimes e_i$

$\mathbb{C} \to \mathbb{C}$
$1 \mapsto \sum_{i=1}^{n} e^i(e_i) = n = \dim V.$

Example: $\mathrm{Rep}_q SL_2\mathbb{C}$ $\quad V \cong V^*$ ⇝ unoriented!

$\sim q^{1/2} \,)\,(\; + q^{-1/2}$, $\bigcirc \sim -q-q^{-1}$

Sanity check: $|| \sim$

$\sim q^{1/2} \; + q^{-1/2} \; \sim q^{1/2}(q^{1/2} \; + q^{-1/2}) + q^{-1/2}(q^{1/2} \; + q^{-1/2})$

$= || + (q+q^{-1}) \; + \; \sim ||$

Intermission: Why Are We Doing This?

- From skein relations, we can define

  Skein algebras $\mathrm{SkAlg}_{\mathcal{A}}(\Sigma)$ Skein modules $\mathrm{SkMod}_{\mathcal{A}}(M)$ 3-manifold

  Skein categories $\mathrm{SkCat}_{\mathcal{A}}(\Sigma)$ surface   Skein functors $\underline{\mathrm{Sk}}_{\mathcal{A}}(M) : \mathrm{SkCat}_{\mathcal{A}}(\partial M)^{op} \longrightarrow \mathrm{Vect}$

- We can recover all these from the skein TFT we're building towards

- When $\mathcal{A} = \mathrm{Rep}\,G$ these recover G-character stacks/varieties $\mathrm{Ch}_G(X) := \mathrm{Hom}(\pi_1(X), G)/\!/G$

  eg $\mathrm{SkCat}_{\mathrm{Rep}G}(\Sigma) \simeq \mathrm{QCoh}(\mathrm{Ch}_G(\Sigma))$

- So, replacing $\mathrm{Rep}\,G \rightsquigarrow \mathrm{Rep}_q\,G$ $(= \mathrm{Rep}\,U_q\mathfrak{g})$ is a form of quantisation!

Topological Field Theories

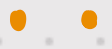
Idea: An assignment $M^n \mapsto Z(M^n)$ with good gluing properties.

Precisely: it's a symmetric monoidal n-functor:

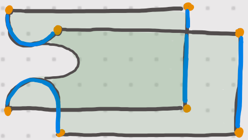
$$Z : Cob^{\sqcup}_{n+k+1,\dots,k} \longrightarrow \mathcal{V}^{\otimes}$$

$\leftarrow$ Choice of n-category

Def: $Cob_{n+k,\dots,k}$:

objects: k-manifolds 

1-morphisms: k+1-manifolds 

2-morphisms: k+2-manifolds with corners 

⋮

n-morphisms: n+k manifolds with corners

Product: Disjoint union.

We'll work with Oriented spaces!

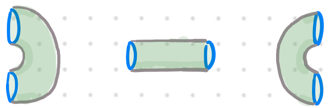
How to make this an n-category?

Sources and targets

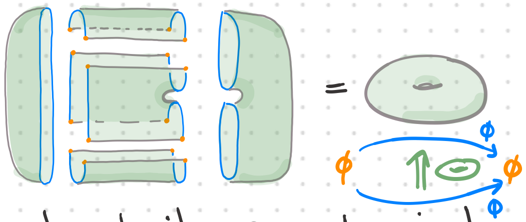
Given by a decomposition of the boundary:

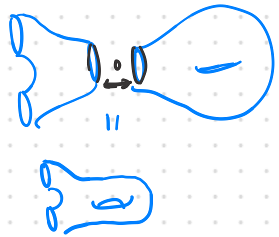
$$M_0 \xrightarrow{W} M_1 \longleftrightarrow \partial W = M_0 \cup \overline{M_1} \cup M_c \leftarrow \text{"the rest"}$$

So: different cobordisms can have the same space:



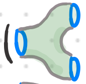
Composition is given by gluing:



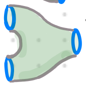


The target inherits Properties & structures:

$Z : Cob_{2,1} \longrightarrow Vect$

$Z($$)$ - product

$Z($$)$ - unit

$Z($$)$ - coproduct

$Z($$)$ - counit

TFTs and Graphical Calculus

Goal: A TFT $\mathrm{Sk}_{\mathcal{A}} : \mathrm{Cob}_{3,2} \longrightarrow \mathrm{Bimod}$

Surfaces ↦ linear Categories; 3-cobordisms ↦ bimodule functors $\mathcal{A} \times \mathcal{B}^{op} \to \mathrm{Vect}$

Composition: by coend!

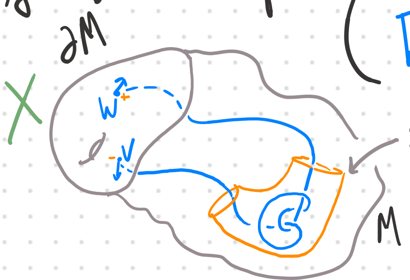
$F : \mathcal{A} \times \mathcal{B}^{op} \longrightarrow \mathrm{Vect}$
$G : \mathcal{B} \times \mathcal{C}^{op} \longrightarrow \mathrm{Vect}$

$$G \circ F(a,c) = \int^{b \in \mathcal{B}} G(b,c) \otimes F(a,b) \simeq \bigoplus_{b \in \mathcal{B}} G(b,c) \otimes F(a,b) \Big/ \sim \text{ when } \exists\, f : b \to b'$$

Def M - a 3-manifold, X - $\mathcal{A}$-labelling on ∂M

the relative skein module is

$$\mathrm{Sk}_{\mathcal{A}}(M; X) := \mathrm{Span}\left\{ \begin{array}{c} \mathcal{A}\text{-coloured} \\ \text{ribbon graphs} \\ \Gamma \hookrightarrow M \end{array} \;\middle|\; \Gamma|_{\partial M} = X \right\} \Big/ \sim$$



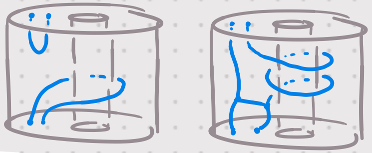
skein relations imposed locally, where $\mathrm{ev}(\Gamma|_{\square})$ is defined

On Surfaces:

Def The Skein Category $\mathrm{SkCat}_{\mathcal{A}}(\Sigma)$ has

objects: $\mathcal{A}$-labellings of Σ

morphisms: $\mathrm{Hom}(X, Y) := \mathrm{Sk}_{\mathcal{A}}(\Sigma \times [0,1]; \bar{X} \sqcup Y)$

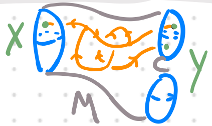


On cobordisms:

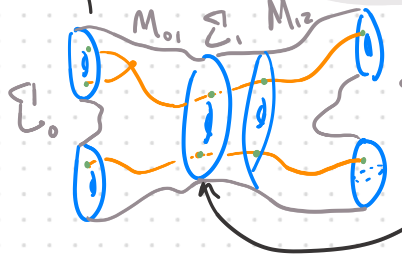
Def Let $\Sigma_0 \xrightarrow{M} \Sigma_1$ be a cobordism.

the skein functor is:

$$\mathrm{Sk}_{\mathcal{A}}(M) : \mathrm{SkCat}_{\mathcal{A}}(\Sigma_0) \times \mathrm{SkCat}_{\mathcal{A}}(\Sigma_1)^{op} \longrightarrow \mathrm{Vect}$$
$$X \qquad Y \longmapsto \mathrm{Sk}_{\mathcal{A}}(M; X \sqcup Y)$$



Composition:

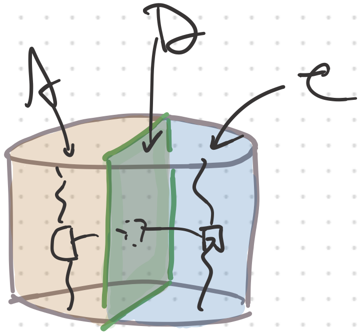


Take colimit over $\mathrm{SkCat}(\Sigma_1)$ to get all skeins!

What's Next?

- Skein Lasanga Modules - Graphical calculus of surfaces in 4d
  - Strong 4-manifold invariants!
  - Knot homology theories!

- Defects - Different theories in different regions



  - Wild Character Varieties (Julia Bierent - Algebra)
  - Spectral Networks (Subrabalan Murugesan - Next!)
  - As (higher) symmetries in gauge theories (Giorgio Pizzolo, Benjamin Haake - Quantum)

Thank You!