GRT Talk 6: Equivariant derived categories

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Plan $G \curvearrowright X$

G alg. group$/\mathbb{C}$, X complex var.

- Notion of "G-equivariant sheaf"

$$Sh_G(X, \mathbb{C})$$

- Generalization to D^bCoh:

$$D^b_G(X, \mathbb{C})$$

Due to issues with $D^bSh_G(X, \mathbb{C})$

i.e. it doesn't "see" topology of G

$$D^b_G(X, \mathbb{C}) \neq D^bSh_G(X, \mathbb{C})$$

$\rightsquigarrow$ Equivariant cohomology

G smooth lin alg group.

i.e. matrix subgroup of $GL(r, \mathbb{C})$

$\sigma : G \times X \longrightarrow X$, X G-Var.

$\mathcal{F} \in Sh(X, \mathbb{C})$ ($\mathbb{C}$-mod valued sheaf)

Lift $G \curvearrowright X$ to $G \curvearrowright \mathcal{F}$?

$g \in G$, $\theta_{g,U} : \mathcal{F}(U) \xrightarrow{\cong} \mathcal{F}(g.U)$

$\theta_{e,U} = id_{\mathcal{F}(U)}$ (Identity)

$h \in G$, $\theta_{gh,U} = \theta_{g,U} \circ \theta_{h,U}$ (Associativity)

Package this as:

$$\theta_g : \mathcal{F} \xrightarrow{\cong} \sigma_g^* \mathcal{F}$$

$$(\sigma_h^* \theta_g) \circ \theta_h = \theta_{gh} \implies \sigma_h^* \sigma_g^* \cong \sigma_{gh}^*$$

Issue Purely group-theoretic, does not account for the topology of G

So θ_g should be a restriction to $\{g\}\times X$ of same isom. in $G\times X$.

$$m, b, pr_{23} : G\times G\times X \longrightarrow G\times X$$

$$m(g,h,x) = (gh, x)$$
$$b(g,h,x) = (g, hx)$$
$$pr_{23}(g,h,x) = (h,x)$$

Def A G-equivariant sheaf $\mathcal{F}\in Sh(X,\mathbb{C})$ is equipped with iso

$$\theta : pr_2^*\mathcal{F} \xrightarrow{\simeq} \sigma^*\mathcal{F}$$

s.t.

$$b^*\theta \circ pr_{23}^*\theta = m^*\theta \quad \text{(Associativity)}$$

A morphism of G-equiv sheaves $\varphi : (\mathcal{F},\theta) \to (\mathcal{G},\tau)$ has

$$\begin{tikzcd} pr_2^*\mathcal{F} \arrow[r, "\theta"] \arrow[d, "pr_2^*\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & \sigma^*\mathcal{F} \arrow[d, "\sigma^*\varphi"] \\ pr_2^*\mathcal{G} \arrow[r, "\tau"] & \sigma^*\mathcal{G} \end{tikzcd}$$

$\rightsquigarrow$ Category $Sh_G(X,\mathbb{C})$ is an abelian category

Now upgrade this to perverse sheaves?

Reminder $D^b(Sh_{constr}(X))$ has perverse

Sheaves that are local sys on stratification

-t- structure (from Ruth's talk).

$D^b_{cst}{}^{\le 0} = \{F \mid \forall i \ \dim supp(H^i(F)) \le -i\}$

$D^b_{cst}{}^{\ge 0} = \{F \mid \forall i \ \dim supp(H^i(DF)) \le -i\}$

$(a : X \to pt)$ (Verdier dual wrt. dualizing cplx $\omega_X = a_X^! \mathbb{Q}_{pt}$ (Scott's talk))

G-equivariance generalizes directly with no issues $\rightsquigarrow$ Obtain $Perv_G(X)$

Note, $\theta : pr_2^* F \xrightarrow{\cong} \sigma^* F$

$$b^*\theta \circ pr_{23}^* \theta = m^* \theta$$

are no longer perverse, but

Shifting by $[\dim G]$ makes e.g. $\sigma^*(F[\dim G])$ perverse (nice since Perv is abelian)

Smooth pullbacks

$\boxed{\text{Prop } 6.2.6, 6.2.7}$

Thm, $f : X \to Y$ smooth surj. morph of G-Vars, then:

$f^! : Perv_G(Y, \mathbb{C}) \to Perv_G(X, \mathbb{C})$ faithful,

f conn fibers $\Rightarrow$ $f^!$ fully faithful & closed under subobj. and quotients

reflects exact sequences

(e.g. $f : \mathbb{A}^2 \setminus 0 \to \mathbb{P}^1 = (\mathbb{A}^2 \setminus 0)/\mathbb{C}^\times$)

Sanity Check $G \curvearrowright X$,

$$\mathrm{Perv}_{\{e\}}(X, \mathbb{C}) \cong \mathrm{Perv}(X, \mathbb{C}) \cong \mathrm{Perv}_G(\tilde{X}, \mathbb{C})$$

(Forgetting functor forgets G-equiv.)

(Inflation functor adds G-equivariance)

$\tilde{X} = X$ but with trivial G-action

Lemma (i) $\pi : X \to X/G$, projection | Prop 6.2.10

$$\pi^! : \mathrm{Perv}(X/G, \mathbb{C}) \xrightarrow{\cong} \mathrm{Perv}_G(X, \mathbb{C})$$

category eq.

(ii) Extension of varieties X H-var. $X(G) = G \times^H X$

$H \leq G$

$i : X \to X(G)$ inclusion, then

$$i^*[-\dim(G/H)] : \mathrm{Perv}_G(X(G), \mathbb{C}) \xrightarrow{\cong} \mathrm{Perv}_H(X, \mathbb{C})$$

Equivariant derived categories

Problem $\mathrm{Perv}(X)$ is the heart of $D^b_{const}(X)$. Is $\mathrm{Perv}_G(X)$ the heart of $D^b_{const, G}(X)$ for some triangulated category?

$= D^b(Sh_{const, G}(X))$?

✗ G-equivariance messes up triangulated cat. properties (if distinguished triangles lifted from $D^b_{const}(X)$)

$= D^b(\mathrm{Perv}_G(X))$, $D^b(\mathrm{Sh}_G(X))$

triangulated, but not nice (e.g. does Lemma before apply?)

$D^b_{c,G}(X,\mathbb{C})$ Wishlist

(1) bounded t-structure s.t. heart is $\mathrm{Perv}_G(X)$ $D^b(\mathrm{Perv}_G(X))$ ✓

(2) $\mathrm{For}^G : D^b_G(X,\mathbb{C}) \to D^b_{const}(X,\mathbb{C})$ is t-exact, restricts to hearts $\mathrm{Perv}_G(X,\mathbb{C}) \longrightarrow \mathrm{Perv}(X,\mathbb{C})$

(3) Six functor formalism $\otimes^L$, $R\mathrm{Hom}$, f^*, f_*, $f_!$, $f^!$ exists wrt. $f: X \to Y$ G-equiv. $\rightsquigarrow D^b_G(X,\mathbb{C})$, $D^b_G(Y,\mathbb{C})$

Example

Ex 6.4.2

$D^b(\mathrm{Perv}_G(X))$ fails (3) $\left(G = GL_2(\mathbb{C}) \circlearrowright \mathbb{P}^1_{\mathbb{C}}\right)$

$X = \mathbb{P}^1_{\mathbb{C}}$, $a: \mathbb{P}^1_{\mathbb{C}} \longrightarrow pt$

a^* is a derived cat. equivalence $D^b_G \mathrm{Perv}(pt) \xrightarrow{\simeq} D^b_G \mathrm{Perv}(\mathbb{P}^1)$

BUT a^* does not admit a right adjoint a_* that is compatible with G-equivariance $\lightning$ (3) (otherwise: $a_* \underline{\mathbb{C}}_{\mathbb{P}^1} = \underline{\mathbb{C}}_{pt}$)

Def X G-variety a G-equivariant complex $\mathcal{F}$ on X is

- $\forall p: P \to X$ G-resolutions, i.e. P principal-G-variety and p G-equiv. (i.e. $G \curvearrowright P$ free),
  $\rightsquigarrow \mathcal{F}(p) \in D^b_{const}(P/G, \mathbb{C})$
- $\forall$ $P \xrightarrow{\nu \text{ equiv}} Q$ $\quad \bar{\nu}: P/G \to Q/G$

$$\begin{tikzcd} P \arrow[rr, "\nu"] \arrow[dr, "p"', "G\text{-res.}"] & \arrow[d, phantom, "\circlearrowright"] & Q \arrow[dl, "q", "G\text{-res}"'] \\ & X & \end{tikzcd}$$

(Morphism of G-resolutions)

have $\alpha_\nu : \bar{\nu}^* \mathcal{F}(q) \xrightarrow{\cong} \mathcal{F}(p)$

Such that

(1) $\alpha_{id} = id_{\mathcal{F}(p)}$

(2) $\alpha_\nu \circ \bar{\nu}^* \alpha_\xi = \alpha_{\xi \circ \nu}$

G-action properties fulfilled

for morph of G-res. ν and ξ

This reminds me of quotient stacks:

$[X/G]$ cat over $Sch/\mathbb{C}$

objects$/S$ are $\begin{tikzcd} P \arrow[r, "G\text{-equiv.}"] \arrow[d, "G\text{-bundle}"'] & X \\ S & \end{tikzcd}$

Think of P as a collection of "G-orbits"

Morph of G-equivariant complexes $\phi: \mathcal{F} \to \mathcal{G}$ have $\forall p$

$$\begin{tikzcd} \bar{\nu}^*\mathcal{F}(q) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F}(p) \arrow[d] \\ \bar{\nu}^*\mathcal{G}(q) \arrow[r] & \mathcal{G}(p) \end{tikzcd}$$

$\rightsquigarrow D^b_{C,G}(X, \mathbb{C})$ category

Ex X G-variety, M fin gen $\mathbb{C}$-mod

$\underline{M}_X$ = equivariant const sheaf

$\underline{M}_X(p) = \underline{M}_P$ for all G-resolutions p

$\alpha_\nu : \nu^* \underline{M}_Q \xrightarrow{\simeq} \underline{M}_P$

canonical isom induced by pullback

Triangulated Structure of $D^b_{c,G}(X,\mathbb{C})$

$pr_2 : G \times X \longrightarrow X$ G-resolution

$G \curvearrowright G \times X$ $g \cdot (h, x) = (hg^{-1}, gx)$

$\mathrm{For} : D^b_G(X,\mathbb{C}) \to D^b_c(X,\mathbb{C})$

$\mathcal{F} \longmapsto \mathcal{F}(pr_2)$

$[1]$ Shift functor

on $D^b_{const}(P/G, \mathbb{C})$

$\forall p : P \longrightarrow X$ G-res.

$(\mathcal{F}[1])(p) = \mathcal{F}(p)[1]$

Obtain distinguished triangles

$\mathcal{F} \longrightarrow \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]$

Define t-structure through

$D^{[a,b]}_G(X,\mathbb{C}) = \{ \mathcal{F} \in D^b_G(X,\mathbb{C}) \mid \mathrm{For}(\mathcal{F}) \in D^b_c(X,\mathbb{C})^{[a,b]} \}$

Verify axioms $\rightsquigarrow$ triangulated structure and t-structure

In particular, For is t-exact triangulated functor. $\rightsquigarrow$ (Wishlist ✓)

Equivariant Cohomology

Def (Equivariant cohomology)

For $\mathcal{F} \in D^b_G(X, \mathbb{C})$

k-th equivariant hypercohomology is

$$\mathbb{H}^k_G(X, \mathcal{F}) = \mathrm{Hom}_{D^b_G(pt,\mathbb{C})}(\underline{\mathbb{C}}_{pt}, a_{X*}\mathcal{F}[k])$$

For cohomology, M fin gen $\mathbb{C}$-mod,

$$H^k_G(X, M) = \mathbb{H}^k_G(X, \underline{M}_X)$$

Fact $\mathcal{F}, \mathcal{G} \in D^b_G(X, \mathbb{C})$

$\exists$ Isom $\mathrm{Hom}(\mathcal{F}, \mathcal{G}[k]) \cong \mathbb{H}_G(X, R\mathcal{H}om(\mathcal{F}, \mathcal{G}))$

Ex $\mathbb{C}^\times \curvearrowright pt$.

$$\mathbb{H}^*_{\mathbb{C}^\times}(pt, \mathbb{C}) \cong \mathrm{Sym}(\mathbb{C})$$

Where ring structure given by $\deg(\mathbb{C}) = 2$.

Ex T torus

Char lattice

$$H^\bullet_T(pt, \mathbb{C}) \cong \mathrm{Sym}(\mathbb{C} \otimes_{\mathbb{Z}} X(T))$$