MOTIVIC HALL ALGEBRAS AND DT INVARIANTS

- goal: introduce Joyce's theory of motivic Hall algebras
  - translate categorical statements into identities in suitable Hall algebras, then apply integration maps to get identities with generating functions for invariants

1 MOTIVIC INVARIANTS

→ meaning invariants such that

$$\chi(X) = \chi(Y) + \chi(U)$$

where $Y \subset X$ closed subvariety and $U = X \setminus Y$

E.g. the Euler characteristic

$$e(X) = \sum_{i \in \mathbb{Z}} (-1)^i \dim H^i(X^{an}, \mathbb{C}) \in \mathbb{Z}$$

- Behrend '09: DT invariants have motivic properties

$\mathcal{M}$ fine projective moduli scheme parametrising stable coherent sheaves on CY3-fold X,

$$DT(\mathcal{M}) = \int_{[\mathcal{M}]^{vir}} 1 \in \mathbb{Z} \quad \text{(degree of the virtual fundamental class)}$$

$\Rightarrow DT(\mathcal{M}) = e(\mathcal{M}, \nu) := \sum_{n \in \mathbb{Z}} n \cdot e(\nu^{-1}(n)) \in \mathbb{Z}$ where $\nu : \mathcal{M} \to \mathbb{Z}$ is a constructible function depending on singularities of $\mathcal{M}$

2 HALL ALGEBRAS

WARM-UP: FINITARY HALL ALGEBRAS

Suppose $\mathcal{A}$ is essentially small abelian s.t.

(i) every object has finitely many subobjects;

(ii) $\mathrm{Ext}^i_{\mathcal{A}}(E,F) < \infty \quad \forall E, F \in \mathcal{A}$

E.g. $\mathcal{A} = A\text{-mod}$ where A is a f.d. algebra over $\mathbb{F}_q$

Def^n. The finitary Hall algebra of $\mathcal{A}$ is the set of all complex-valued functions on iso classes of $\mathcal{A}$

$$\widehat{\mathrm{Hall}}_{fty}(\mathcal{A}) = \{ f : (\mathrm{Ob}(\mathcal{A})/\sim) \longrightarrow \mathbb{C} \}$$

with convolution product

$$(f_1 * f_2)(B) = \sum_{A\subset B}' f_1(A)\cdot f_2(B/A) \qquad (\text{i.e. } 0\to A\to B\to B/A\to 0)$$

$\to$ associative, usually non-commutative, unital (= characteristic function δ_0 of the zero object)

- $\mathrm{Hall}_{fty}(\mathcal{A}) \subset \widehat{\mathrm{Hall}}_{fty}(\mathcal{A})$ is the subalgebra of functions w/ finite support
- for $E\in\mathcal{A}$, let $\delta_E \in \mathrm{Hall}_{fty}(\mathcal{A})$ be the characteristic function of iso class of E and define $\kappa_E = |\mathrm{Aut}(E)|\cdot\delta_E \in \mathrm{Hall}_{fty}(\mathcal{A})$

<u>Lem.</u> For any objects $A, C\in\mathcal{A}$ we have

$$\kappa_A * \kappa_C = \sum_{B\in\mathcal{A}}' \frac{|\mathrm{Ext}^1(C,A)_B|}{|\mathrm{Hom}(C,A)|}\cdot\kappa_B$$

where $\mathrm{Ext}^1(C,A)_B \subset \mathrm{Ext}^1(C,A)$ is the subset of extensions whose middle term is isomorphic to B.

- define $\delta_{\mathcal{A}} \in \widehat{\mathrm{Hall}}_{fty}(\mathcal{A})$ by $\delta_{\mathcal{A}}(E) = 1 \quad \forall E\in\mathcal{A}$
- fix $P\in\mathcal{A}$,
  - $\delta_{\mathcal{A}}^P \in \widehat{\mathrm{Hall}}_{fty}(\mathcal{A})$, $\quad \delta_{\mathcal{A}}^P(E) = |\mathrm{Hom}_{\mathcal{A}}(P,E)|$
  - $\mathrm{Quot}_{\mathcal{A}}^P \in \widehat{\mathrm{Hall}}_{fty}(\mathcal{A})$, $\quad \mathrm{Quot}_{\mathcal{A}}^P(E) = |\mathrm{Hom}_{\mathcal{A}}^{\twoheadrightarrow}(P,E)|$ where $\mathrm{Hom}_{\mathcal{A}}^{\twoheadrightarrow}(P,E)\subset\mathrm{Hom}_{\mathcal{A}}(P,E)$ is the subset of surjective maps

<u>Lem.</u> There is an identity

$$\delta_{\mathcal{A}}^P = \mathrm{Quot}_{\mathcal{A}}^P * \delta_{\mathcal{A}} \in \widehat{\mathrm{Hall}}_{fty}(\mathcal{A}).$$

<u>Pf.</u> Evaluating for $E\in\mathcal{A}$ yields

$$|\mathrm{Hom}_{\mathcal{A}}(P,E)| = \sum_{A\subset E}' |\mathrm{Hom}_{\mathcal{A}}^{\twoheadrightarrow}(P,A)|\cdot 1,$$

which holds because every map $f: P\to E$ factor uniquely through its image. ∎

<u>GENERAL IDEA</u>

- different types of Hall algebras should be thought of as different ways to take the „cohomology" of the moduli stack of objects of an abelian category $\mathcal{A}$

(take $\mathcal{A} = \mathrm{Coh}(X)$ on smooth projective variety X)

- consider $\mathcal{M}$ the stack of objects of $\mathcal{A}$, and $\mathcal{M}^{(2)}$ the stack of short exact sequences

$$\mathcal{M}\times\mathcal{M} \xleftarrow{(a,c)} \mathcal{M}^{(2)} \xrightarrow{b} \mathcal{M} \qquad (*)$$

  - map (a,c) is of finite type, but not representable

- map b is representable, but only locally finite type
- the fibre of (a,c) over $(A,C) \in \mathcal{M} \times \mathcal{M}$ is the quotient stack

$$[\mathrm{Ext}^1_X(C,A) / \mathrm{Hom}_X(C,A)]$$

- the fibre of b over $B \in \mathcal{M}$ is the Quot scheme $\mathrm{Quot}_X(B)$

Goal: find suitable „cohomology theory" for our stacks and use (*) to get a product

$$m: H^*(\mathcal{M}) \otimes H^*(\mathcal{M}) \longrightarrow H^*(\mathcal{M})$$

i.e. a rule that assigns a vector space to each stack s.t.

(a) for every morphism of stack $f: X \to Y$, there are functorial maps

$$f^*: H^*(Y) \longrightarrow H^*(X), \qquad f_*: H^*(X) \longrightarrow H^*(Y)$$

when f is of finite type or representable, respectively

(b) there are functorial Künneth maps

$$H^*(X) \otimes H^*(Y) \longrightarrow H^*(X \times Y)$$

MOTIVIC HALL ALGEBRA

Def$^{\underline{n}}$. The relative Grothendieck group $K(Sch/S)$ is the free abelian group on the set of iso classes of S-schemes $f: X \to S$ (X is finite type over $\mathbb{C}$) modulo

$$[X \xrightarrow{f} S] \sim [Y \xrightarrow{f|_Y} S] + [U \xrightarrow{f|_U} S],$$

this condition ensures that the group is not trivial

where $Y \subset X$ closed subscheme and $U = X \setminus Y$

↳ fibre product over S gives a ring structure

- if $\varphi: S \to T$ is a map of schemes,

$$\Rightarrow \quad \varphi_*: K(Sch/S) \longrightarrow K(Sch/T)$$
$$[f: X \to S] \mapsto [\varphi \circ f: X \to T]$$

$$\Rightarrow \quad \varphi^*: K(Sch/T) \longrightarrow K(Sch/S) \qquad \text{if } \varphi \text{ is finite type}$$
$$[g: Y \to T] \mapsto [g \times_T S: Y \times_T S \to S]$$

and $[f: X \to S] \otimes [g: Y \to T] \mapsto [f \times g: X \times Y \to S \times T]$

Def$^{\underline{n}}$. The motivic Hall algebra is the relative Grothendieck group

$$\mathrm{Hall}_{mot}(\mathcal{A}) := K(St/\mathcal{M})$$

with product

$$[Y_1 \xrightarrow{f_1} \mathcal{M}] * [Y_2 \xrightarrow{f_2} \mathcal{M}] = [Z \xrightarrow{b\circ h} \mathcal{M}] \quad \text{given by}$$

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}^{(2)} \arrow[r, "b"] \arrow[d] & \mathcal{M} \\ Y_1 \times Y_2 \arrow[r, "f_1\times f_2"] & \mathcal{M}\times\mathcal{M} & \end{tikzcd}$$

- an elt of the Hall algebra ≈ family of objects of $\mathcal{A}$ over some base stack, and the product of two families is given by taking their universal extension

3 INTEGRATION MAPS

= homomorphism from $\mathrm{Hall}_{\mathrm{mot}}(\mathcal{A})$ into (skew) polynomial rings (using integration of cohomology class over the moduli space)

- let $\mathcal{A}$ be a k-linear abelian category and s.t. $\dim \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i_{\mathcal{A}}(A,B) < \infty \quad \forall A,B\in\mathcal{A}$

↳ Euler form: $\chi(-,-): K_0(\mathcal{A}) \times K_0(\mathcal{A}) \longrightarrow \mathbb{Z}$ defined by

$$\chi(E,F) = \sum_{i\in\mathbb{Z}} (-1)^i \dim \mathrm{Ext}^i(E,F)$$

- fix a character map $ch: K_0(\mathcal{A}) \longrightarrow N$, where N is the charge lattice = free abelian grp of finite rank

satisfying:
- Euler form descends to a bilinear form $\chi(-,-): N\times N \to \mathbb{Z}$
- $ch(E)$ is locally constant on families

↳ $N \cong \mathbb{Z}^{\oplus n}$ ⇒ define the quantum torus algebra for $(-,-)$

$$\mathbb{C}_t[N] = \bigoplus_{\alpha\in N} \mathbb{C}(t)\cdot x^\alpha, \quad \text{with} \quad x^\alpha * x^\gamma = t^{-(\gamma,\alpha)} \cdot x^{\alpha+\gamma}$$

HEREDITARY CASE ($\mathrm{Ext}^i_{\mathcal{A}}(E,F)=0$ for $i>1$)

Lem [Reineke]. When $\mathcal{A}$ is hereditary, there is an algebra homomorphism

$$\mathcal{I}: \mathrm{Hall}_{\mathrm{fty}}(\mathcal{A}) \longrightarrow \mathbb{C}_t[N]|_{t=\sqrt{q}}, \qquad \mathcal{I}(f) = \sum_{E\in\mathcal{A}} \frac{f(E)}{|\mathrm{Aut}(E)|}\cdot x^{ch(E)}$$

whose codomain is the quantum torus for $2\chi(-,-)$.

- similar result for motivic case: $\mathcal{M} = \bigsqcup_{\alpha\in N} \mathcal{M}_\alpha$ virtual Poincaré invariant

$$\mathcal{I}: \mathrm{Hall}_{\mathrm{mot}}(\mathcal{A}) \longrightarrow \mathbb{C}_q[N], \qquad \mathcal{I}([S\to\mathcal{M}_\alpha]) = \chi_t(S)\cdot x^\alpha$$

CY3 CASE

$\mathcal{A} = \mathrm{Coh}(X)$, X cpx projective CY3-fold ⇝ Euler form is skew-symmetric

Kontsevich-Soibelman: $\mathcal{I}: \mathrm{Hall}_{\mathrm{mot}}(\mathcal{A}) \longrightarrow \mathbb{C}_t[N]$ (quantum torus for Euler form)

- uses motivic vanishing cycles?

Bridgeland: introduce the semi-classical limit of $\mathbb{C}_t[N]$ at $t=\varepsilon$

$\Rightarrow$ commutative Poisson algebra

$$\mathbb{C}_\varepsilon[N] = \bigoplus_\gamma \mathbb{C}\cdot x^\gamma \quad \text{with} \quad x^\alpha \cdot x^\gamma = \lim_{t\to\varepsilon}(x^\alpha * x^\gamma) = \varepsilon^{-(\alpha,\gamma)}\cdot x^{\alpha+\gamma}$$

$$\{x^\alpha, x^\gamma\} = \lim_{t\to\varepsilon}\left(\frac{x^\alpha * x^\gamma - x^\gamma * x^\alpha}{t^2-1}\right) = (\alpha,\gamma)\cdot x^\alpha x^\gamma$$

$\Rightarrow$ semi-classical limit of motivic Hall algebra $\mathrm{Hall}_{sc}(\mathcal{A})$ which is a commutative Poisson algebra

$\Rightarrow$ $I_\varepsilon : \mathrm{Hall}_{sc}(\mathcal{A}) \longrightarrow \mathbb{C}_\varepsilon[N_t]$

$$I_\varepsilon([S \xrightarrow{f} \mathcal{M}_\alpha]) = \begin{cases} e(S)\cdot x^\alpha & \text{if } \varepsilon = +1 \\ e(S; f^*(\nu))\cdot x^\alpha & \text{if } \varepsilon = -1 \end{cases}$$

$\longrightarrow$ naive DT invariants

$\longrightarrow$ actual DT invariants

Th$^{\underline{m}}$ [Toda, Bridgeland].

(i) For each $\beta \in H_2(X,\mathbb{Z})$ we have

$$\sum_{n\in\mathbb{Z}}' PT(\beta,n)y^n = \frac{\sum'_{n\in\mathbb{Z}} DT(\beta,n)y^n}{\sum'_{n\in\mathbb{Z}} DT(0,n)y^n}$$

(ii) This formal power series is the Laurent expansion of a rational function of y, invariant under $y \leftrightarrow y^{-1}$.