Torus localization $T = \mathbb{C}^*$

Goal. A generic cubic surface in $\mathbb{P}^3$ contains exactly 27 lines.

Strategy. $\Sigma \subset \mathbb{P}^3$ cubic hypersurface

$$f \in \mathbb{C}[x_0, x_1, x_2, x_3]_3 \subset H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$$

Σ contains a line ℓ iff the restriction map

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) \longrightarrow H^0(\ell, \mathcal{O}_\ell(3))$$

vanishes.

Let $G(1,3)$ be the Grassmanian of lines in $\mathbb{P}^3$. Tautological line bundle $S \to G(1,3)$.

$\text{Lines}(\Sigma) \subset G(1,3)$.

Consider the bundle

$$\text{Sym}^3 S^* \longrightarrow G(1,3)$$

Fiber over $\ell \in G(1,3)$ = space of cubic functions on ℓ.

From Σ get a section

$$f_\Sigma \in H^0\left(G(1,3), \text{Sym}^3 S^*\right)$$

by restricting Σ to each line, giving a cubic.

We have: $\text{Zero}\left(f_\Sigma\right) = \text{Lines}(\Sigma)$

$\subseteq G(1,3)$.

By general algebraic topology, we have that

$$[\mathrm{Zero}(T_\Sigma)] \in H^0(\mathbb{G}(1,3))$$

is Poincare dual (if Σ generic) to

$$e(\mathrm{Sym}^3 S^*) = c_4(\mathrm{Sym}^3 S^*)$$
$$\in H^8(\mathbb{G}(1,3)).$$

We have:

$$\#\text{ lines in } \Sigma = \int_{\mathbb{G}(1,3)} e(\mathrm{Sym}^3 S^*).$$

Goal: <u>compute this integral.</u>

<u>Next strategy</u>: $\mathbb{C}^\times \curvearrowright \mathbb{P}^3$

$$\begin{array}{c} \mathbb{C}^\times \\ \| \\ \mathbb{T} \end{array}$$

$$t \cdot [x_0 : x_1 : x_2 : x_3] = [t^{w_0} x_0 : t^{w_1} x_1 : t^{w_2} x_2 : t^{w_3} x_3].$$

$$w_0, w_1, w_2, w_3 \in \mathbb{Z}.$$

$\rightarrow$ induces an action on $G(1,3)$

And this will induce an obvious action on $Sym^3(S^*)$.

$$\int_{G(1,3)} e^{\mathbb{C}^\times}\left(Sym^3(S^*)\right).$$

$\rightarrow$ will localize to the fixed points of the action.

$\rightarrow$ will allow us to compute this integral.

$$\sum_{[\ell] \in \mathcal{G}(1,3)^{\mathbb{C}^\times}} \frac{e^{\mathbb{C}^\times}\left(\mathrm{Sym}^3 S^*|_\ell\right)}{e^{\mathbb{C}^\times}\left(T_\ell \mathcal{G}(1,3)\right)}$$

fixed lines of $\mathbb{C}^\times$-action

§ Equivariant cohomology.

Let G be a topological group, X a (right) G-space, $\rho : G \times X \to X$.

Naive thing to consider:

$$H^*_{\mathrm{sing}}(X/G).$$

- if G acts freely, this is a reasonable thing.
- if not, cf. $G \supset \mathbb{R}$.

Idea: find a contractible space EG with a free right G-action

and consider

$$EG \times^G X = \frac{EG \times X}{(eg, x) \sim (e, gx)}$$

l.h.

And define

$$H^*_G(X) := H^*_{sing}(EG \times^G X).$$

Fact. Such an EG always exists.

Examples: $G = \mathbb{Z}$

$EG = \mathbb{R}$, $\mathbb{Z} \curvearrowright \mathbb{R}$ by translations

$\mathbb{R}/\mathbb{Z} = S^1$.

$G = \mathbb{C}^\times$

$EG = (\mathbb{C}^\infty) \setminus \{0\}$

$EG/\mathbb{C}^\times = \mathbb{P}^\infty$.

Generally $EG/G =$ "BG" "classifying space of G"

Because $\forall$ X nice enough space X

$$\left\{ \begin{array}{c} \text{principal } G\text{-bundles} \\ P \to X \end{array} \right\}/\text{iso} \longleftrightarrow \left\{ \begin{array}{c} \text{maps} \\ X \to BG \end{array} \right\}/\text{htpy.}$$

$$\begin{array}{ccc} f^*EG & \to & EG \\ \downarrow & & \downarrow \\ X & \to & BG \\ & f & \end{array}$$

Examples of equivariant cohomology

1. $X = pt$. $G \curvearrowright pt$ trivially.

Then $H^*_G(pt) = H^*(BG)$.

For $\mathbb{C}^\times$: $H^*_{\mathbb{C}^\times}(pt) = H^*(\mathbb{P}^\infty; \mathbb{Q})$

$= \mathbb{Q}[S]$

S in degree 2

S is $c_1\left(\mathcal{O}_{\mathbb{P}^\infty}(-1)\right)$.

2. if G acts freely on X, then

$$X \times^G EG \longrightarrow X/G$$

fibers are EG, contractible.

$\rightsquigarrow H_G^*(X) \cong H^*(X/G)$.

3. If G acts trivially on X,

$$H_G^*(X;\mathbb{Q}) = H^*(X) \otimes_{\mathbb{Q}} H_G^*(pt).$$

One trick for computation

theorem. If E is some space with a right free G-action such that

$$\pi_i(E) = 0 \text{ for all } i \leq m.$$

then

$$H^i(E \times^G X) \cong H_G^i(X)$$

for all $i \leq m$.

Example of this: $G = \mathbb{C}^\times$.

$$(\mathbb{C}^n)^{ss}(\mathbb{C}) \longrightarrow \mathbb{P}^n.$$

$$S^{2n-1}$$

So if you want to compute

$$H^i_{\mathbb{C}^\times}(X)$$

can use $\mathbb{C}^i(\mathbb{C}) \longrightarrow \mathbb{P}^i$

Equivariant vector bundles and classes

Def. Let X be a left G-space.

An equivariant vector bundle over X

is:

1) A vector bundle $E \xrightarrow{\pi} X$

2) A lift of the G-action on E, i.e.

a map $G \times E \xrightarrow{act} E$ s.t.

$$\begin{array}{ccc} G \times E & \xrightarrow{act} & E \\ \downarrow{\scriptstyle id \times \pi} & \circlearrowright & \downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow[act]{} & X \end{array}$$

Construction: if $E \to X$ is an equivariant vector bundle, consider

$$V_E : EG \times^G E \to EG \times^G X$$

same rank as E.

$$c_i(V_E) \in H^{2i}(EG \times^G X)$$

sq

$$H^{2i}_G(X).$$

Def. The equivariant chern classes of E are given by

$$c_i^G(E) := c_i(V_E) \in H^{2i}_G(X).$$

From now on $G = \mathbb{C}^*$.

Easiest example of equivariant vector bundle:

$E \longrightarrow X$. data: $\begin{cases} \text{vector space } V \\ \mathbb{C}^* \to \mathrm{End}(V) \end{cases}$

So this is just a $\mathbb{C}^\times$-representation.

say $\operatorname{rank} V = 1$. $\quad \mathbb{C}_a \to pt$

$t \cdot v = t^a \cdot v$ for some $a \in \mathbb{Z}$.

Can try to compute $c_1^{\mathbb{C}^\times}(\mathbb{C}_a)$

Pick a model of $E\mathbb{C}^\times$:

$$\mathbb{C}^2 \setminus \{0\} \longrightarrow \mathbb{P}^\infty.$$

To compute $c_1^{\mathbb{C}^\times}(\mathbb{C}_a)$, I get the bundle

$$\mathbb{C}^2 \setminus \{0\} \times^{\mathbb{C}^\times} \mathbb{C}_a \longrightarrow \mathbb{C}^2 \setminus \{0\} /_{\mathbb{C}^\times} = \mathbb{P}^\infty$$

$$c_1^{\mathbb{C}^\times}(\mathbb{C}_a) := c_1\left(\mathcal{O}_{\mathbb{P}^\infty}(-a)\right)$$

$\in H^*_{\mathbb{C}^\times}(pt) = \mathbb{Q}[s]$. s deg 2

a - S.

More generally: if $V \to pt$. of finite rank, then

$$V = \bigoplus_{i=1}^{\operatorname{rank} V} \mathbb{C}_{a_i} \qquad a_i \in \mathbb{Z}.$$

$$e^{\mathbb{C}^\times}(V) = \prod_{i=1}^{\operatorname{rank} V} c_1^{\mathbb{C}^\times}(\mathbb{C}_{a_i})$$

$$= a_1 a_2 \cdots a_{\operatorname{rank} V}\, s^{\operatorname{rank} V} \in \mathbb{Q}[s].$$

More generally:

Let $E \to X$ but $\mathbb{C}^\times$ acts trivially.

Then $E = \bigoplus_{a \in \mathbb{Z}} E_a$

(not necessarily rank 1).

To compute chern classes: associated bundles.

$$V_{E_a} \longrightarrow X \times \mathbb{P}^\infty$$

$$\cong$$

check $E_a \boxtimes \mathbb{C}_a$

$$= \pi_X^*(E_a) \otimes \pi_{\mathbb{P}^\infty}^*(\mathbb{C}_a).$$

$$\boxed{c_i^{\mathbb{C}^\times}\left((NC)_a\right) = \sum_{k=0}^{i} \underbrace{\binom{\operatorname{rank} E_a - k}{i-k} c_k(NC_a)\, a^{i-k}}_{\uparrow} \otimes \underbrace{s^{i-k}}_{2(i-k)}}$$

in: $H^*_{\mathbb{C}^\times}(C) \overset{\text{action trivial}}{=} \underbrace{H^*(C) \otimes \mathbb{Q}[s]}_{2k}.$

if $\epsilon > 0$ $= (\text{nilpotent}) + \binom{\operatorname{rank} E_a}{i} a^i s^i$

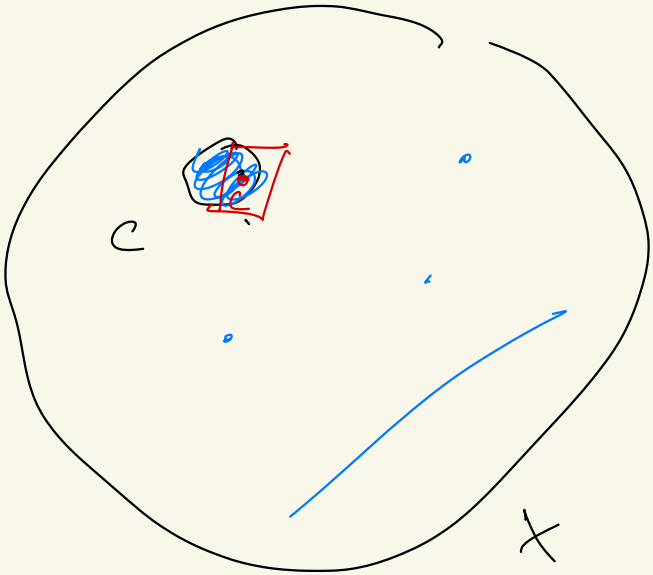
§ Torus fixed loci

$\mathbb{C}^\times \curvearrowright X$, X smooth.

$X^{\mathbb{C}^\times}$ = fixed point set of action.

TX has an induced $\mathbb{C}^\times$ action from $\mathbb{C}^\times \curvearrowright X$.

C component of $X^{\mathbb{C}^\times}$



Facts: let $c \in C \subset X^{\mathbb{C}^\times}$.

Then $(T_c X)^{\mathbb{C}^\times} = T_c C$.

$\Rightarrow$ the induced action on

$$N_c C = T_c X / T_c C$$

is trivial, i.e. $(N_c C)^{\mathbb{C}^\times} = 0$.

We have

$$NC \longrightarrow C$$

1. $\mathbb{C}^\times$ acts trivially on C.

Thus $NC = \bigoplus_{a \neq 0} (NC)_a$.

2. None of the a's are zero

$\Rightarrow$ $c_i((NC)_a)$ is invertible

in $H^*_{\mathbb{C}^\times}(C)[s^{-1}]$.

$\Rightarrow$ $e(NC)$ invertible in $H^*_{\mathbb{C}^\times}(C)[s^{-1}]$.

The localization theorem

Non-equivariant setting.

If $f: M^m \to N^n$ is a smooth map of oriented manifolds, can construct

$f_!: H^*(M) \to H^{*+\operatorname{codim}(f)}(N)$.

"integration map"

→ what gives rise to integration of forms, by looking at $\varphi : M \to pt$.

$$\varphi_!(\alpha\text{-form}) =: \int_M (\alpha\text{-form}).$$

These maps can be corrected in equivariant cohomology as well.

localisation theorem

Let $\mathbb{C}^\times \curvearrowright X$, let $X^{\mathbb{C}^\times} \xhookrightarrow{i} X$ fixed point locus.

Then

$$i_* : H^*_{\mathbb{C}^+}\left(X^{\mathbb{C}^+}\right) \longrightarrow H^*_{\mathbb{C}^+}(X)$$

is an isomorphism after inverting S.

Formula for the inverse:

$$\psi \in H^*_{\mathbb{C}^+}(X)[S^{-1}]$$

$$\longmapsto \sum_{C_\alpha \in X^{\mathbb{C}^+}} \frac{i^*\psi}{e^{\mathbb{C}^+}(N C_\alpha)}$$

invertible in $H^*_{\mathbb{C}^+}(C_\alpha)[S^{-1}]$.

Integration formula for the case when the fixed point locus $X^{\mathbb{C}^\times}$ is a finite set of points, we have that for $\psi \in H^*_{\mathbb{C}^\times}(X)[S^{-1}]$ we have

$$\int_X \psi = \sum_{q \in X^{\mathbb{C}^\times}} \frac{i_q^* \psi}{e^{\mathbb{C}^\times}(T_q X)}$$

$$\in H^*_{\mathbb{C}^\times}(\text{pt})[S^{-1}].$$

Application.

Make $\mathbb{C}^*$ act on $\mathbb{P}^3$ via

$$t \cdot [x_0 : x_1 : x_2 : x_3] = (t^{w_0} x_0 : t^{w_1} x_1 : t^{w_2} x_2 : t^{w_3} x_3)$$

for distinct integers w_0, w_1, w_2, w_3.

Fixed points of this action:

$[1:0:0:0]$ $[0:1:0:0]$ $[0:0:1:0]$

$[0:0:0:1]$.

Call them p_i

Fixed lines: l_{ij} connecting p_i & p_j

$$l_{ij} = \mathbb{C}x_i^* \oplus \mathbb{C}x_j^* \subset G(1,3).$$

$$G(1,3)^{\mathbb{C}^\times} = \{ l_{ij} \mid 0 \le i < j \le 3 \}.$$

$$\int_{G(1,3)} e\left(Sym^3 S^* \right)$$

$$= \int_{G(1,3)} e^{\mathbb{C}^\times}\left(Sym^3 S^* \right).$$

(localization)

$$= \sum_{l_{ij}} \frac{e^{\mathbb{C}^\times}\left(Sym^3 S^* / l_{ij} \right)}{e^{\mathbb{C}^\times}\left(T_{l_{ij}} G(1,3) \right)}.$$

Fact:

$$T_\ell G(1,3) \cong \mathrm{Hom}(E, V/E).$$

$$V = \mathrm{Span}_{\mathbb{C}}(x_0^*, x_1^*, x_2^*, x_3^*)$$

coordinates on $\mathbb{C}^4$

A line is $E \subset V$ of dimension 2

For us: $\ell_{ij} \rightsquigarrow \mathbb{C}x_i^* \oplus \mathbb{C}x_j^*$

So V/E is $\mathbb{C}x_k^* \oplus \mathbb{C}x_\ell^*$

where $\{i,j,k,\ell\} = \{0,1,2,3\}$.

So $\mathrm{Hom}(E, V/E) = (\mathbb{C}x_i \oplus \mathbb{C}x_j) \otimes (\mathbb{C}x_k^* \oplus \mathbb{C}x_\ell^*)$

$= E^\vee \otimes V/E$

$$= \mathrm{span}_{\mathbb{C}}\Big(\underbrace{t_i \otimes t_k^*}_{w_i - w_k},\ \underbrace{t_j \otimes t_k^*}_{w_j - w_k},$$

$$t_i \otimes t_k^*,\ t_j \otimes t_k^* \Big)$$

$$w_i - w_k \qquad w_j - w_k.$$

$$e^{C^+}\Big(T_{l_{ij}}\, G(1,3) \Big) =$$

$$= (w_i - w_k)(w_j - w_k)(w_i - w_k)$$

$$(w_j - w_k) \cdot s^4$$

$$\in H_{\mathbb{C}^*}^{8}(pt)[s^{-1}].$$

$$\mathrm{Sym}^3 S^* / l_{ij} = \mathrm{Sym}^3\Big(\mathbb{C}t_i \oplus \mathbb{C}t_j \Big)$$

$$= \mathbb{C}\Big(t_i^3,\ t_i^2 t_j,\ t_i t_j^2,\ t_j^3 \Big).$$

$$e^{\mathbb{C}^*}\left(S_{\mathrm{gm}}{}^3 S^*/_{\ell_{ij}}\right)$$

$$= 3w_i \cdot (2\cdot w_i + w_j)(w_i + 2w_j)(3w_j)\cdot s^4$$

$$\in H^8_{\mathbb{C}^*}(\mathfrak{p})[s^{-1}].$$

$$\sum_{ij} \frac{3w_i(2w_i+w_j)(w_i+2w_j)(3w_j)}{(w_i-w_k)(w_j-w_i)(w_j-w_k)(w_i-w_k)}$$

$$\in \mathbb{Z}$$

this is 27.