Virtual Cycles:

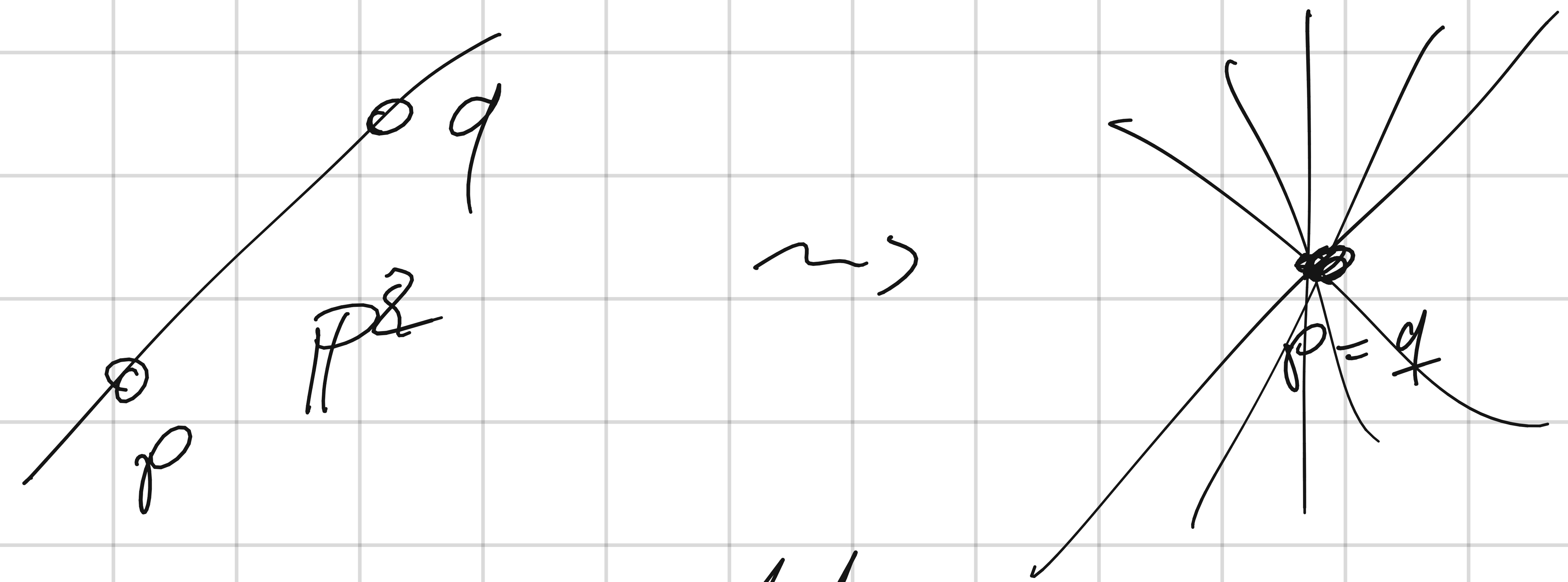
Problem: Occasionally, a counting problem does not have the "right" dim:

$$\int_{[M]} 1 = 0 \quad , \text{as } \dim_{\mathbb{C}} [M] \neq 0$$

Ex: Let $N_{p,q}$ the # of lines $\ell \subset \mathbb{P}^2$ through 2 points $p, q \in \mathbb{P}^2$.

$$N_{p,q} = \begin{cases} 1 & \text{for } p \neq q \\ \infty & \text{for } p = q \end{cases}$$

Where $N_{p,q} = \infty$ corresponds to the cardinality of the pencil $Z_p \cong \mathbb{P}^1$ of lines through $p \in \mathbb{P}^2$.



$\Rightarrow$ We want to obtain $N_{p,q}^{\text{corrected}} = 1$ somehow!

Our problem should not depend on minor perturbations, we are reasonable people.

Defn: ((Co)normal sheaf) Given a closed immersion $\iota : X \hookrightarrow M$ defines an ideal $\mathcal{I} \subset \mathcal{O}_M$. The q.coh sheaf of $\mathcal{O}_X$-modules

pullback along ι^*

$$\mathcal{C}_{X/M} = \mathcal{I}/\mathcal{I}^2$$ is the conormal sheaf of X in M.

$$\mathcal{N}_{X/M} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$$ is the normal sheaf of X in M.

$\mathcal{N}_{X/M}$, $\mathcal{C}_{X/M}$ are locally free of rank d when $X \hookrightarrow M$ is regular of codim d.

Ex: $\iota : X \hookrightarrow \mathbb{P}^r$ a hypersurface of degree d, then the ideal sheaf of X in $\mathbb{P}^r$ is $\mathcal{O}_{\mathbb{P}^r}(-d)$, so $N_{X/\mathbb{P}^r} = \mathcal{O}_{\mathbb{P}^r}(d)|_X$

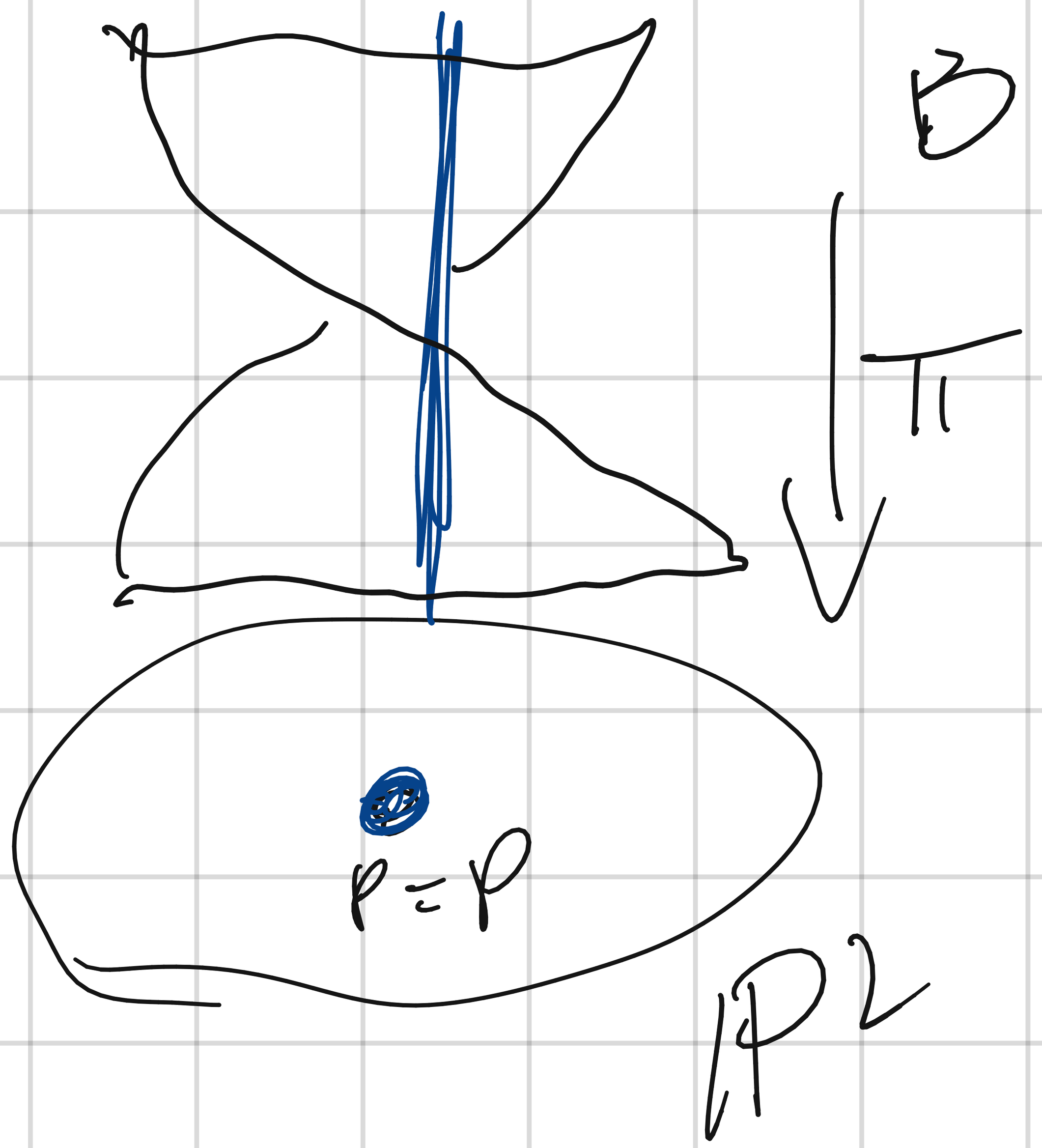
Rmk: Recall that on any $\mathbb{P}^r$ we have the Euler sequence:

$$0 \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r}(1)^{\oplus r+1} \to T_{\mathbb{P}^r} \to 0$$

w/ $\mathcal{O}_{\mathbb{P}^r}(1)$ hyperplane bun, $T_{\mathbb{P}^r}$ alg. tangent bun.

Now go back to our original problem:

$$\pi : B \to \mathbb{P}^2$$



$E \cong \mathbb{P}^1$ is the exceptional divisor of π.

Hence:

$$\begin{tikzcd}
\mathbb{P}^1 = E \arrow[r, hook] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\pi"] \\
\mathbb{P}^2 \ni p \arrow[r, hook] & \mathbb{P}^2
\end{tikzcd}$$

and thus $N_{E/B} = \mathcal{O}_E(-1)$

$N_{E/B} \subset f^* N_{p/\mathbb{P}^2}$

We can define the obstruction bundle:

$Ob_{p,p} \to \mathbb{P}^1$ via the quotient:

$$0 \to \mathcal{O}_E(-1) \to \mathcal{O}_E \otimes T_p\mathbb{P}^2 \to Ob_{p,p} \to 0$$

Therefore: $Ob_{p,p} := \mathcal{T}_{\mathbb{P}^1} \otimes \mathcal{O}_{E=\mathbb{P}^2}(-1) = \mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^2}(-1) = \mathcal{O}_{\mathbb{P}^1}(1)$

Note: If $p \neq q$, then $Ob_{p,q} = 0$ $\left(\pi^{-1}(q) = pt \neq \mathbb{P}^1\right)$

If $M_{p,q} = \pi^{-1}(q)$ is the "moduli space" of lines $l \subset \mathbb{P}^2$ passing through $p,q \in \mathbb{P}^2$, then the *virtual* number of $\# l$ is

$$\int_{[M_{p,q}]} e(Ob_{p,q}) = 1$$

$$\Longrightarrow N^{\text{connected}}_{p,q} = 1 \quad \forall p,q \in \mathbb{P}^2.$$

<u>Upshot</u>: Take a non-transverse problem and try to remove excess words. (End of trivial introduction).

<u>Story</u>: The <u>virtual</u> fund. class allows us to think of M (associated Moduli space of counting problem) even if "horrible", we still have associated a counting problem in virtual dimension d^{virt} at any point $p \in M$.

$$\text{"} d^{virt} = \dim T_p M - \dim Ob_p$$

where Ob is part of the data p, o, t defining $[M]^{virt}$.

Heuristics: $[M]^{virt} \in A_{d^{virt}}(M)$

encoding the deformation theory of points $p \in M$ in the sense that

$M|_p$ is cut out by $\dim Ob_p$ # equations in $\dim T_pM$ # variables.

- smooth, unobstructed, there are no equations "$Ob = 0$".
- smooth, obstructed, if the equations are linearly independent "$Ob_p \neq 0$ is a bundle"
- horrible, when $Ob \neq 0$ not locally free!

Cases that work theory:

1) M is smooth, then $[M]^{virt} = e(Ob) \cap [M]$

$$\int_{[M]^{virt}} \alpha = \deg_M (e(Ob) \cap \alpha), \alpha \in A^*(M)$$

2) M is the zero locus of a section of a Vec bun

$\pi : V \to A$, $\iota : M \hookrightarrow A$, A smooth projective.

Then $\iota_* [M]^{virt} = e(V) \cap [A]$.

3) M admits a torus action $\Rightarrow$ virtual localization

Let Y smooth var. of $\dim_{\mathbb{C}} Y = d$ and let $\operatorname{Spec} \operatorname{Sym} \mathcal{E}^{\vee}$ be the total space of a rank r V.bun on Y. Furthermore, $s \in H^0(Y, E) = \operatorname{Hom}_Y(\mathcal{O}_Y, \mathcal{E})$ a section and $\mathcal{I} = \operatorname{im}(s^{\vee}) \subset \mathcal{O}_Y$ be the ideal sheaf of the vanishing locus

$$X = Z(s) \subset Y.$$

$$\begin{tikzcd}
X \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "s"] \\
Y \arrow[r, "0"'] & E
\end{tikzcd}$$

then $d^{\text{virt}} = d - r$

$$\Rightarrow [X]^{\text{virt}} \in A_{d-r}(X)$$

The image ann $(s^\vee)$ of the cosection $\left(s^\vee : \mathcal{E}^\vee \to \mathcal{O}_Y \text{ is } \mathcal{I}\right)$

hence we get

$$b : \mathcal{E}^\vee\big|_X \twoheadrightarrow \mathcal{I}/\mathcal{I}^2$$

Applying the Spec Sym construction we obtain the closed immersion

$$\operatorname{Spec}\operatorname{Sym} \mathcal{I}/\mathcal{I}^2 = \underbrace{N_{X/Y}}_{\text{Normal of } X \text{ in } Y.} \hookrightarrow E\big|_X$$

Define $C_{X/Y} = \operatorname{Spec}\operatorname{Sym} \bigoplus_n \mathcal{I}^n/\mathcal{I}^{n+1}$, then $C_{X/Y} \hookrightarrow N_{X/Y}$

is again a closed immersion of the normal cone in normal bundle.

$$\begin{tikzcd} C_{X/Y} \arrow[r, hook] & N_{X/Y} \arrow[r, hook] & E|_X \arrow[d] \\ & & X \arrow[u, bend right, "0"'] \end{tikzcd}$$

where $C_{X/Y}$ is a purely d-dim subvar. of $E|_X$.

Recall $0^* : A_d(E|_X) \xrightarrow{\sim} A_{d-r}(X)$.

<u>Defn</u>: The virt fund. cycle of $X = Z(s) \subset Y$ is the Chow class

$$[X]^{\text{virt}} = 0^*[C_{X/Y}] \in A_{d-r}(X).$$

Defn: (Truncated cotangent complex) The cotangent sheaf Ω_X of a scheme X is the 0-th cohomology of a canonical object

$$\mathbb{L}_X \in D^{[-1,0]}(X).$$

Ex: If X quasi-proj, and if $\mathcal{I} \subset \mathcal{O}_Y$ is the ideal of the inclusion $i : X \hookrightarrow Y$ then

$$\mathbb{L}_X = [\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} \Omega_Y|_X]$$

$\Rightarrow$ does not depend on embedding.

Defn: A complex $\mathbb{E}$ of $\mathcal{O}_X$-modules is p.c.p.a in $[a,b]$ if it is locally iso to a complex

$$[E^a \to \dots \to E^b], \text{ locally free, finite rank.}$$

Defn: (p.o.t) A p.o.t. of a scheme X is a pair $[\mathbb{E}, \phi]$ where $\mathbb{E}$ is a p.c.p.a in $[-1,0]$ and $\phi : \mathbb{E} \to \mathbb{L}_X$ is a morphism $D^{[-1,0]}(X)$:

1) $h^0(\phi)$ is an iso

2) $h^{-1}(\phi)$ is surj.

$d^{\text{virt}} = \mathrm{rk}(\mathbb{E})$

$\mathrm{rk}(E^0) - \mathrm{rk}(E^{-1})$ if $\mathbb{E}$ is locally $[E^{-1} \to E^0]$

<u>Defn</u>: A p.o.t $(\mathbb{E}, \phi)$ is symmetric if

$$\theta : \mathbb{E} \xrightarrow{\sim} \mathbb{E}^\vee[1]$$

s.t. $\theta = \theta^\vee[1]$.

<u>Rmk</u>: If $(\mathbb{E}, \phi)$ is symmetric, then $Ob = \Omega_X$. In particular $h^{-1}(\phi) = Ob^\vee = T_X$.

Rmk:

$$\begin{tikzcd} \mathbb{E} = [\mathcal{E}^\vee|_X \arrow[r, "d\circ s"] \arrow[d, "\phi"'] \arrow[d, shift left=6, "\sigma"] & \Omega_X|_X] \arrow[d, "id"] \\ \mathbb{L}_X = [\mathcal{I}/\mathcal{I}^2 \arrow[r, "d"] & \Omega_Y|_X] \end{tikzcd}$$

$\Rightarrow$ 1) If $X = Z(s) \subset Y$, then

$$Ob = \mathrm{coker}(N_{X/Y} \hookrightarrow \mathcal{E}|_X)$$

2) If $\mathcal{E} = \Omega_Y$, $s = df \in H^0(Y, T^* Y)$ for a regular function, $X = \mathrm{crit}(f)$

$$\mathbb{E} = [d\circ\sigma : T_Y|_X \to \Omega_Y|_X]$$

and $d^{virt} = 0$, $\mathbb{E} = \mathrm{Hess}(f)$, intrinsic to X.

<u>Rmk</u>: $\mathrm{Hilb}^n(\mathbb{A}^3)$ carries a s.p.o.t.

<u>Thm</u>: [G.P] $\quad \iota : X^{fix} \hookrightarrow X \qquad X \circlearrowleft \mathbb{C}^\times$

$$[X]^{virt} = \iota_* \sum_i \frac{[X_i]^{virt}}{e^{\mathbb{C}^\times}(N_i^{virt})}$$

- $X \hookrightarrow Y$, $\bigsqcup_i Y_i = Y^{fix}$, $X_i = X \cap Y_i$, X carries a p.o.t.
- $N_i^{virt} = \mathbb{E}_i^{mov} \in D^{[0,1]}(\mathrm{Coh}^{\mathbb{C}^\times} X_i)$
- $[X]^{virt} \in A_*^{\mathbb{C}^\times}(X) \otimes_{\mathbb{Q}[t]} \mathbb{Q}(t)$, t a $\mathbb{C}^\times$ character.

Appendix:

$$[\overline{M}_{g,\emptyset,(d)}(X/D,\beta)]^{virt}, \quad \mathbb{P}^1 = X, \quad D = \{\infty\}$$

genus g curves, NO marked points, maximal tangency profile

$$\beta = H_2(\mathbb{P}^1, \mathbb{Z})$$

$$\left\{\int_{[M]^{virt}} e^{\mathbb{C}^*}(Ob)\right\} = \int_{[\overline{M}_{g,1}(\mathbb{P}^1,d)]} \iota^*\left[\frac{e(Ob)}{e(N^{virt})}\right] = \left\{\int_{[\overline{M}_{g,1}]} (-1)^g \lambda_g \psi^{2g-2}\right\}$$

$$d^{2g-2}\frac{B_{2g}}{(2g-2)!\,2g}$$