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Enumerative Geometry Talk 4: relative Quot and Hilb

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Relative Grassmannian, short recap:

How? $\Rightarrow \underbrace{\mathcal{M} : Sch^{op} \longrightarrow Set}_{\text{representable functor}}$ (Sch, noetherian)

First off, every S-scheme $\mathcal{M}$ trivially "represents" its own functor of points,

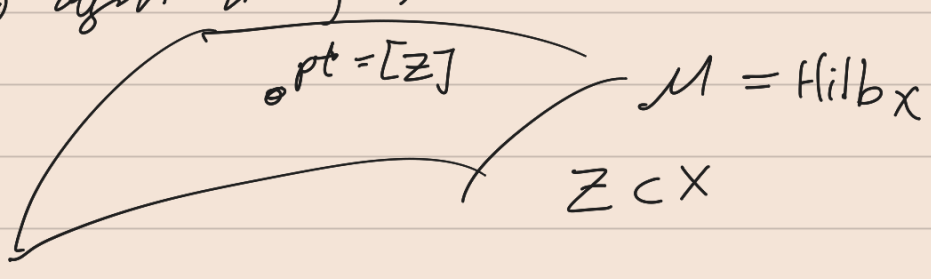
$$\text{namely}: \quad h_{\mathcal{M}} : Sch_S^{op} \longrightarrow Set$$

$$U \mapsto h_{\mathcal{M}}(U) = Hom_S(U, \mathcal{M}).$$

"Let $\mathcal{T}$ a class of "things", and there is a family of things defined over $U \in Sch_S$ and suppose you can pull back such family along arbitrary maps $V \to U$, $U, V \in Sch_S$. Then $\mathcal{M}$ is a fine Moduli space for objects in $\mathcal{T}$, if the $\underline{\mathcal{M}}$ assigning to a scheme U the set of "families of objects in $\mathcal{T}$ defined over U"

$$h_{\mathcal{M}} = Hom(-, \mathcal{M}).$$

Ex: "things" $\mathcal{T} \rightsquigarrow Z \subset X$ Z closed subscheme in X.
Each point in this fine Moduli space $\mathcal{M}$ is parameterized precisely by objects in $\mathcal{T}$:



relation Quot & Hilb :

Fix S a noetherian and $X \to S$ of finite type (hence X is also noetherian).
Furthermore fix a coherent sheaf $\mathcal{F}$ on X.

Def: (Quot)

Given $U \in Sch_S$ define

$$Quot_{X/S}(\mathcal{F})(U \to S)$$

to be the set of equivalence classes

$$(\mathcal{E}, p),$$

where:
- $\mathcal{E}$ a coherent sheaf on $X \times_S U$, flat over U and whose support is proper over U.
- $p : \mathcal{F}_U \twoheadrightarrow \mathcal{E}$ is an $\mathcal{O}_{X \times_S U}$-linear surjection where w/ $\mathcal{F}_U$ we mean the pullback along

$$X \times_S U \to X$$

- $(\mathcal{E}, p) \sim (\mathcal{E}', p') \iff \ker p = \ker p'$

Rmk: We get properness of $\operatorname{supp}(\mathcal{E}_U) \hookrightarrow X \times_S U \downarrow U$ for free when $X \to S$ is projective.

Def: (Holomorphic Euler characteristic / Hilb polynomial)

Fix a field k and a very ample line bundle L over X/k. For a coherent sheaf E on X w/ $\mathrm{supp}(E)$ proper over $\mathrm{Spec}\, k$, define the function:

$$m \longmapsto P_L(E,m) := \chi(E \otimes_{\mathcal{O}_X} L^{\otimes m})$$

$\Rightarrow$ is a polynomial for large $m \gg 0$.

$$\chi(E) := \sum_{i \geq 0} (-1)^i \dim_k H^i(X, E).$$

$\rightsquigarrow$ relative Setting:

If $\mathcal{E}$ is an S-flat family of coherent sheaves on $X \to S$ s.t.

$\mathrm{Supp}(\mathcal{E}) \hookrightarrow X \to S$ is proper, then:

$$s \longmapsto P_{L_s}(\mathcal{E}_s), \quad \mathcal{E}_s := \mathcal{E}|_{X_s}$$

is locally constant on S, where L_s denotes the restriction along $X_s \hookrightarrow X$, $s \in S$.

If $P_{L_s}(\mathcal{E}_s)$ constant along the fibers $s \in S$ then this implies S-flatness of the family $\mathcal{E}$ (if S is reduced).

$\rightsquigarrow$ local constancy can be checked over closed points.

For a fixed (relatively) ample line bundle $L \in \mathrm{Pic}(X)$, $X \to S$, the functor $\mathrm{Quot}_{X/S}(\mathcal{F})$ decomposes into components labeled by $(P_L(\mathcal{F}) \in \mathbb{Q}[z])$

$$\wr$$

$$\mathcal{O}_Y(z)$$

$$\left[\mathrm{Quot}_{X/S}(\mathcal{F}) = \bigsqcup_{P_L \in \mathbb{Q}[z]} \mathrm{Quot}^{P_L}_{X/S}(\mathcal{F}) \right] !$$

where each stratum $\mathrm{Quot}^{P_L}_{X/S}(\mathcal{F}) : \mathrm{Sch}^{op}_S \to \mathrm{Set}$ sends a scheme U to the set of equivalent classes of quotients:

$$p : \mathcal{F}_U \twoheadrightarrow \mathcal{E}$$

s.t. $\forall u \in U$, $P_L(\mathcal{E}_u)$ (whose support is a closed subscheme of X_u, proper over $\mathrm{Spec}\, k$) calculated via the ample bundle L_u (pullback of relatively ample bundle L along $X_u \hookrightarrow X \times_S U \to X$.

Thm: (Grothendieck) S noetherian. If $X \to S$ projective, $\mathcal{F} \in \mathrm{Coh}(X)$, L a relatively very ample line bundle over X and $P_L \in \mathbb{Q}[z]$ a polynomial, then

$$\mathrm{Quot}^{P_L}_{X/S}(\mathcal{F})$$

is representable by a projective S-scheme.

Consequence: $X \hookrightarrow \mathbb{P}^n_S$ a closed subscheme, $L = \mathcal{O}_{\mathbb{P}^n_S}(1)|_X$ and $\mathcal{F}$ a quotient

$$p: \mathcal{F} \twoheadrightarrow L(m)^{\oplus l} = \left(\mathcal{O}_{\mathbb{P}^n_S}(m)|_X\right)^{\oplus l}$$

for (m, l) then:

$$\left[\mathrm{Quot}^{P_L}_{X/S}(\mathcal{F}) \hookrightarrow \mathbb{P}^N_S\right] \text{ is an embedding}$$

for some N (large enough).

(If X a hypersurface $\subset \mathbb{P}^n$ for example, then $N = \binom{n+m}{n}$)

Def: (Quot scheme of points)

$X \to S$, $n \in \mathbb{Z}_{\geq 0}$, $\mathcal{F} \in \mathrm{Coh}(X)$, then we define

$$\mathrm{Quot}_{X/S}(\mathcal{F}, n) := \mathrm{Quot}^{n_L}_{X/S}(\mathcal{F}) \subset \mathrm{Quot}_{X/S}(\mathcal{F})$$

where the choice of $L \in \mathrm{Pic}(X)$ is irrelevant as the Hilbert polynomial is constant $P_L(\mathcal{F}) = n$. If $\mathcal{O}_X = \mathcal{F}$ we call the above object

$$\mathrm{Hilb}^n_{X/S} \left(= \mathrm{Hilb}^n(X/S)\right) := \mathrm{Quot}_{X/S}(\mathcal{O}_X, n) \subset \mathrm{Hilb}_{X/S}$$

Rmk: For any locally free sheaf $\mathcal{L}$, w/ $\mathrm{rk}\,\mathcal{L} = 1$ over a k-variety X, we get the isomorphism $\mathrm{Quot}_{X/k}(\mathcal{L}, n) = \mathrm{Hilb}^n(X)$.

Plane conics Example:

Let z_0, z_1, z_2 be homogeneous coord of $\mathbb{P}^2/\mathbb{C}$. And let $a_0, \dots, a_5$ be — " — coord of $\mathbb{P}^5$. Consider the closed subscheme

$$C \subset \mathbb{P}^2 \times \mathbb{P}^5$$

cut out by the bihomogeneous equation

$$\left[C = \{ a_0 z_0^2 + a_1 z_1^2 + a_2 z_2^2 + a_3 z_0 z_1 + a_4 z_0 z_2 + a_5 z_1 z_2 = 0 \} \subset \mathbb{P}^2 \times \mathbb{P}^5. \right]$$

Let $\pi : C \to \mathbb{P}^5$. Over a point $a = [a_0 : \dots : a_5] \in \mathbb{P}^5$

the fiber is the conic:

$$\pi^{-1}(a) = \{ \dots \} \subset \mathbb{P}^2.$$

Then there is a set-theoretic bijection between $\mathbb{P}^5$ and $\mathrm{Hilb}^{P_C}(\mathbb{P}^2)$. $\rightsquigarrow$ By universality of $\mathbb{P}^n$, one hence scheme-theoretic iso

$$\mathbb{P}^5 \cong \mathrm{Hilb}^{P_C}(\mathbb{P}^2),$$

and $\pi : C \to \mathbb{P}^5$ is the universal family of the Hilb scheme of plane conics!

Note: C of degree d in $\mathbb{P}^2$, and $\deg(\mathcal{O}_C(t)) = d \cdot t$. But $\chi(\mathcal{O}_C(t)) = d \cdot t + 1 - g$.

But g for a conic is 0, and $d = 2$, hence:

$$P_L(t) = 2t + 1.$$

$$\mathbb{P}^S \cong Hilb^{2t+1}(\mathbb{P}^2).$$