HILBERT SCHEME OF n POINTS

Recall that in general,

- X projective scheme, $\mathcal{O}_X(1)$ ample line bundle

$$\mathcal{H}ilb_X : \underline{Sch} \longrightarrow \underline{Set}$$

$$\mathcal{H}ilb_X(U) := \left\{ Z \subset U \times X \;:\; \begin{matrix} Z \text{ closed subscheme,} \\ \begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[d, "\pi"'] & U \times X \arrow[d, "p_2"] \\ U \arrow[r, equal] & U \end{tikzcd} \end{matrix} \;:\; \pi \text{ is flat} \right\}$$

$\Leftrightarrow$ for $u \in U$, $P_u(m) = \chi(\mathcal{O}_{\pi^{-1}(u)} \otimes \mathcal{O}_X(m))$

Th$^{\underline{m}}$ (Grothendieck). The functor $\mathcal{H}ilb_X^P$ is representable by a projective scheme Hilb_X^P.

↳ fine moduli problem, i.e. $\exists$ a universal family $\mathcal{Z}$ on Hilb_X^P

- if $P(m) = n$ $\forall m \in \mathbb{Z}$ $\Rightarrow$ Hilbert scheme of n points

↳ for $Z = \overset{\text{distinct}}{\{x_1, \dots, x_n\}} \subset X$, $\mathcal{O}_Z \otimes \mathcal{O}_X(m) = \mathcal{O}_Z$ $\forall m \in \mathbb{Z}$

„Moduli space of n points on X"

Explicitly,

length of a closed subscheme

$$\mathrm{Hilb}^n(X) = \{ \mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z : \dim H^0(Z, \mathcal{O}_Z) = n \}$$

Def$^{\underline{n}}$. Let S^nX be the symmetric product of X,

$$S^nX = \underbrace{X \times \dots \times X}_{n\text{-times}} / S_n$$

$\Rightarrow$ parametrises effective 0-dimensional cycles of degree n

$$\textstyle\sum_i n_i [x_i] \in S^nX, \quad n_i \in \mathbb{N} \text{ with } \sum_i n_i = n$$

We have the Hilbert-Chow morphism

$$\pi : \mathrm{Hilb}^n(X) \longrightarrow S^nX$$

$$Z \longmapsto \sum_{x \in \mathrm{Supp}\, Z} \mathrm{length}(Z_x)[x]$$

Exmp. $\mathrm{Hilb}^n(\mathbb{C}) \cong S^n\mathbb{C}$ Q: How are they different?

Take $\mathrm{Hilb}^2(\mathbb{C}^2)$, $Z = \{x_1, x_2\}$

↳ suppose $x_1 = x_2$; a vector $\underline{v} \neq 0 \in T_xX$ defines an ideal

$$I = \{ f \in \mathbb{C}[x,y] : f(x) = 0,\ df_x(\underline{v}) = 0 \} \quad (\text{i.e. } m/m^2)$$

$\Rightarrow$ I has length 2 $\rightsquigarrow$ $\mathcal{O}_{\mathbb{C}^2}/I$ gives a 0-dim subscheme

$\rightsquigarrow$ $\mathrm{Hilb}^2(\mathbb{C}^2) = \mathrm{Bl}_\Delta(\mathbb{C}^2 \times \mathbb{C}^2)/S_2$

Th$^{\underline{m}}$ (Fogarty). Suppose X is nonsingular and $\dim X = 2$, then the following holds.

(1) $\mathrm{Hilb}^n(X)$ is nonsingular, and has dimension $2n$.

(2) $\pi : \mathrm{Hilb}^n(X) \longrightarrow S^nX$ is a resolution of singularities.

$\Big\langle$ Ext^1 computations

smoothable computations

HILBERT SCHEME OF POINTS ON AFFINE SPACE

$$\mathrm{Hilb}^n(\mathbb{C}^d) = \{ I \subset \mathbb{C}[x_1, \dots, x_d] : \dim \mathbb{C}[x_1, \dots, x_d]/I \}$$

- ideal $I \in \mathrm{Hilb}^n(\mathbb{C}^d)$ identifies a closed subscheme $\mathrm{Spec}\, \mathbb{C}[x_1, \dots, x_d]/I \subset \mathbb{C}^d$ of length n

$$\Rightarrow \quad \mathbb{C}[x_1, \dots, x_d] \twoheadrightarrow \mathbb{C}[x_1, \dots, x_d]/I$$

- to construct a point in the Hilbert scheme:

  ① vector space $V_n \cong \mathbb{C}^n$ $\quad (\dim V_n = n)$

  ② $\mathbb{C}[x_1, \dots, x_d]$-module structure

$$\vartheta : \mathbb{C}[x_1, \dots, x_d] \longrightarrow \mathrm{End}(V_n)$$

such that such structure is induced by an $\mathbb{C}[x_1, \dots, x_d]$-linear surjection from $\mathbb{C}[x_1, \dots, x_d]$

$\Rightarrow$ ϑ gives matrices $X_1, \dots, X_d \in \mathrm{End}(V_n)$

If we have an $\mathbb{C}[x_1, \dots, x_d]$-linear quotient $\varphi : \mathbb{C}[x_1, \dots, x_d] \twoheadrightarrow V_n$, $\varphi(1)$ must generate V_n as an $\mathbb{C}[x_1, \dots, x_d]$-module

$$\Rightarrow \text{ for any } \underline{w} \in V_n, \quad \underline{w} = X_1^{m_1} X_2^{m_2} \dots X_d^{m_d} \cdot \varphi(1) \quad \text{ for some } m_i \in \mathbb{Z}_{\geq 0}$$

$\Rightarrow$ we specify a cyclic vector $\underline{v} \in V_n$

Consider $\quad W_n = \mathrm{End}(V_n)^{\oplus d} \oplus V_n \quad \Rightarrow \quad GL_n \curvearrowright W_n, \quad g \cdot (X_1, \dots, X_d, v) = (gX_1g', \dots, gX_dg', g \cdot v)$

$\qquad\qquad\ \ \cup$

$\qquad\quad U_n = \{(X_1, \dots, X_d, v) : v \text{ cyclic}\}$ $\qquad$ ↳ this action is free on U_n

Def$^{\underline{n}}$. The GIT quotient

$$\mathrm{ncHilb}^n(\mathbb{C}^d) = U_n /\!/ GL_n$$

Th$^{\underline{m}}$. There is a closed immersion

$$\mathrm{Hilb}^n(\mathbb{C}^d) \hookrightarrow \mathrm{ncHilb}^n(\mathbb{C}^d)$$

cut out by the relations $[X_i, X_j] = 0$.

HILB^n($\mathbb{C}^2$) AS A QUIVER VARIETY

- let $Q = (Q_0, Q_1)$ be a (finite) quiver with $V = (V_i, \{x_h\})_{i \in Q_0, h \in Q_1}$ with dimension vector $\underline{v}$
- $\mathrm{Rep}(Q, \underline{v}) = \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{s(h)}, V_{t(h)})$ and $GL_{\underline{v}} = \prod_{i \in Q_0} GL(V_i)$

  ↳ $GL_{\underline{v}} \curvearrowright \mathrm{Rep}(Q, \underline{v})$ by conjugation

$$\left\{ \begin{array}{l} \text{iso classes of} \\ \text{semisimple } Q\text{-reps} \\ \text{with dim. vector } \underline{v} \end{array} \right\} \longleftrightarrow \mathrm{Rep}(Q, \underline{v}) /\!/ GL_{\underline{v}}$$

- doubled $\bar{Q}$: - for every $h \in Q_1$, add arrow $\bar{h} : t(h) \longrightarrow s(h)$

  - $\bar{Q}_1 = \{\bar{h} : h \in Q_1\}$ $\Rightarrow$ $\bar{Q} = (Q_0, Q_1 \sqcup \bar{Q}_1)$

$$\Rightarrow \quad \mathrm{Rep}(\bar{Q}, \underline{v}) = \bigoplus_{h \in Q_1 \sqcup \bar{Q}_1} \mathrm{Hom}(V_{s(h)}, V_{t(h)}) = \mathrm{Rep}(Q, \underline{v}) \oplus \mathrm{Rep}(Q^{op}, \underline{v})$$

- we have $\mathrm{Rep}(Q^{op},\underline{v}) \cong \mathrm{Rep}(Q,\underline{v})^\vee$ using the canonical perfect pairing

cotangent bundle

$$\Rightarrow \mathrm{Rep}(\bar{Q},\underline{v}) = \mathrm{Rep}(Q,\underline{v}) \oplus \mathrm{Rep}(Q,\underline{v}) = T^*(\mathrm{Rep}(Q,\underline{v}))$$

↳ has a symplectic form $\omega((x_h,x_{\bar{h}}),(y_h,y_{\bar{h}})) = \sum'_{h\in Q_1} \mathrm{tr}(x_h y_{\bar{h}} - y_h x_{\bar{h}})$, which is $GL_{\underline{v}}$-invariant

We have the associated moment map

this is the Lie algebra of $GL_{\underline{v}}$

$$\mu : T^*(\mathrm{Rep}(Q,\underline{v})) \longrightarrow \bigoplus_{i\in Q_0} \mathrm{End}(V_i) = \mathfrak{g}_{\underline{v}} = \mathfrak{g}_{\underline{v}}^*$$

sends $x \in \mathfrak{g}_{\underline{v}}$ to a linear function $y \mapsto \mathrm{tr}(x\cdot y)$

$$(x,y) \longmapsto \sum'_i (xy - yx)$$

Exmp. $Q =$  $\Rightarrow \bar{Q} =$ 

- let $\underline{v} \in \mathbb{Z}$ $\Rightarrow \mathrm{Rep}(\bar{Q},\underline{v}) = \mathfrak{g}_{\underline{v}} \times \mathfrak{g}_{\underline{v}}$

$\Rightarrow GL_{\underline{v}} \curvearrowright \mathrm{Rep}(\bar{Q},\underline{v})$ is the $\mathrm{Ad}\,GL_{\underline{v}}$-diagonal action on pairs of $(\underline{v}\times\underline{v})$-matrices

Aside: $\mu : T^*X \to \mathfrak{g}^*$, $\underset{\substack{\uparrow\\ \text{covector at } x\in X}}{\alpha_x} \mapsto \mu(\alpha_x)$ defined by $\mathfrak{g}^* \ni \mu(\alpha_x) : u \mapsto \langle \alpha, \vec{u}_x \rangle$

- $u \in \mathfrak{g}_{\underline{v}}$, we have $\vec{u} = \mathrm{ad}\,u$

$\rightsquigarrow \mu(x,y) : \mathfrak{g}_{\underline{v}} \to \mathbb{C}$, $u \mapsto \langle y, \vec{u}_x \rangle = \langle y, \mathrm{ad}\,u(x)\rangle = \mathrm{tr}(y\cdot[u,x]) = \mathrm{tr}([x,y]\cdot u)$

$\Rightarrow$ applying $\mathfrak{g}_{\underline{v}}^* \cong \mathfrak{g}_{\underline{v}}$, we get $\mu : \mathfrak{g}_{\underline{v}} \times \mathfrak{g}_{\underline{v}} \to \mathfrak{g}_{\underline{v}}$, $(x,y) \mapsto [x,y]$

not necessary

Def$^{\text{n}}$. For any $\lambda = (\lambda_i)_{i\in Q_0} \in \mathbb{C}^{Q_0}$, let $\Pi_\lambda := \Pi_\lambda(\bar{Q})$ be a quotient of the path algebra $\mathbb{C}\bar{Q}$ by the two-sided ideal generated by

$$\sum'_{h\in Q_1} (h\cdot\bar{h} - \bar{h}\cdot h) - \sum'_{i\in Q_0} \lambda_i \cdot 1_i \in \mathbb{C}\bar{Q}.$$

This is the preprojective algebra of Q with parameter λ.

$$\Rightarrow \mathrm{Rep}(\Pi_\lambda,\underline{v}) = \mu^{-1}(\lambda) := \{(x,y)\in \mathrm{Rep}(\bar{Q},\underline{v}) : [x,y] = \lambda\}$$

Rem. For a rep V of Π_λ,

$$\lambda\cdot\underline{v} = \sum'_{i\in Q_0} \lambda_i \cdot \mathrm{tr}(1_i) = \mathrm{tr}\left(\sum \lambda_i 1_i\right) = \mathrm{tr}\left(\sum'_{h\in Q_1} (h\bar{h} - \bar{h}h)\right) = 0$$

- framed quiver $Q^\heartsuit$: - fix dimension vector $(\underline{v},\underline{w}) \in \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0}$
  - $W = \bigoplus_{i\in Q_0} W_i$ with dimension vector $\underline{w}$
  - add arrows $x_i : W_i \to V_i$

$$\Rightarrow \mathrm{Rep}(\overline{Q^\heartsuit},\underline{v},\underline{w}) = T^*(\mathrm{Rep}(Q^\heartsuit,\underline{v},\underline{w})) = \mathrm{Rep}(Q,\underline{v}) \times \mathrm{Rep}(Q,\underline{v})^\vee \times \mathrm{Hom}_{\mathbb{C}Q_0}(V,W) \times \mathrm{Hom}_{\mathbb{C}Q_0}(W,V)$$

↳ moment map $\mu : \mathrm{Rep}(\overline{Q^\heartsuit},\underline{v},\underline{w}) \longrightarrow \mathfrak{g}^*$

$$(x,y,i,j) \longmapsto \sum'_i [x,y] + ij$$

$= \sum'_{i\in Q_0} i_i j_i$

Def$^{\text{n}}$. Fix a stability condition $\theta \in \mathbb{R}^{Q_0}$ s.t. $\theta\cdot\underline{v} = 0$. The variety $\mathcal{M}_{\lambda,\theta} := \mu^{-1}(\lambda)/\!\!/_{\chi_\theta} GL_{\underline{v}}$ is the Nakajima quiver variety with parameters (λ,θ).

<u>Exmp</u>. $Q =$  , $\underline{v} \in \mathbb{Z}$

$\Rightarrow$ moment map for Q, $\mu : \mathfrak{gl}_{\underline{v}} \times \mathfrak{gl}_{\underline{v}} \longrightarrow \mathfrak{gl}_{\underline{v}}$, $(x,y) \mapsto [x,y]$

so $\mu^{-1}(\lambda \cdot \mathrm{id}) = \{ (x,y) \in \mathfrak{gl}_{\underline{v}} \times \mathfrak{gl}_{\underline{v}} : [x,y] = \lambda \cdot \mathrm{id} \}$

↳ since $\mathrm{tr}([x,y]) = 0$, $\mu^{-1}(\lambda \cdot \mathrm{id}) = \emptyset$ for $\lambda \neq 0$

↳ $\mathfrak{X} := \mu^{-1}(0)$ is the commuting variety of $\mathfrak{gl}_{\underline{v}}$

Let $i : \mathbb{C}^{\underline{v}} \hookrightarrow \mathfrak{gl}_{\underline{v}}$ be the imbedding of diagonal matrices

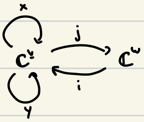
$\Rightarrow$ $i \times i : \mathbb{C}^{\underline{v}} \times \mathbb{C}^{\underline{v}} \longrightarrow \mathfrak{X}$ (since diagonal matrices commute)

(permutation matrices)
$S_{\underline{v}} \subset GL_{\underline{v}}$ act diagonally on $\mathfrak{X} \subset \mathfrak{gl}_{\underline{v}} \times \mathfrak{gl}_{\underline{v}}$ and preserve image of $i \times i$

$\Rightarrow$ restriction of $\mathrm{Ad}\,GL_{\underline{v}}$-invariant functions induces $(i \times i)^* : \mathbb{C}[\mathfrak{X}]^{\mathrm{Ad}GL_{\underline{v}}} \to \mathbb{C}[\mathbb{C}^{\underline{v}} \times \mathbb{C}^{\underline{v}}]^{S_{\underline{v}}}$

$\Rightarrow$ $\mathcal{M}_{0,0}(\underline{v}) := \mu^{-1}(0) /\!/ GL_{\underline{v}} = \mathrm{Spec}\,\mathbb{C}[\mathfrak{X}]^{\mathrm{Ad}GL_{\underline{v}}} = \mathrm{Spec}\,\mathbb{C}[\mathbb{C}^{\underline{v}} \times \mathbb{C}^{\underline{v}}]^{S_{\underline{v}}} = \mathbb{C}^{\underline{v}} \times \mathbb{C}^{\underline{v}} / S_{\underline{v}}$

On the other hand,

$\bar{Q}^{\heartsuit} =$ 

<u>Prop</u>. The canonical projection

$\pi : \mathcal{M}_{\lambda,\theta} \to \mathcal{M}_{\lambda,0}$ is a symplectic resolution.

<u>Claim</u>: For $\underline{v} = n$ and $\underline{w} = 1$, we have $\mu^{-1}(0)^{\theta\text{-ss}} / GL_n \cong \mathrm{Hilb}^n(\mathbb{C}^2)$.

Outline: This matches the explicit description of $\mathrm{Hilb}^n(\mathbb{C}^2)$ with i the cyclic vector, $j = 0$

↳ <u>Prop</u>. A $(n,1)$-dimensional $\mathbb{C}\bar{Q}$-module is θ-semistable $\Leftrightarrow$ it is cyclic

<u>HILB^n $\mathbb{C}^3$ AS A CRITICAL LOCUS</u>

Th$^{\underline{m}}$. There exists a smooth quasiprojective variety M_n with a regular function $f_n : M_n \to \mathbb{C}$ such that

$$\mathrm{Hilb}^n(\mathbb{C}^3) \subset M_n$$

can be realised as as the scheme-theoretic vanishing locus of the exact 1-form df_n.

<u>Sketch of proof</u>. Consider $\tilde{f}_n : W_n \longrightarrow \mathbb{C}$

$$(X_1, X_2, X_3, v) \longmapsto \mathrm{tr}\, X_1 [X_2, X_3]$$

$\Rightarrow$ $\tilde{f}_n(X_1, X_2, X_3) = \sum_{ijk}^{i} (X_1)_{ij} (X_2)_{jk} (X_3)_{ki} - \sum_{i,j,k}^{i} (X_1)_{ij} (X_3)_{jk} (X_2)_{ki}$

↳ differentiating w.r.t. $(X_1)_{ij}$ gives the (j,i) entry of $[X_2, X_3]$

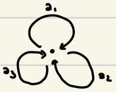
$\Rightarrow$ $\mathrm{Crit}(\tilde{f}_n) = U_n = \{ (X_1, X_2, X_3, v) : [X_i, X_j] = 0, \langle X_1, X_2, X_3 \rangle v = V \}$

$\wr$

$f_n : nc\mathrm{Hilb}^n(\mathbb{C}^3) \longrightarrow \mathbb{C}$ since $\tilde{f}_n$ is GL-invariant

$(X_1, X_2, X_3, v) \longmapsto \mathrm{tr}\, X_1 [X_2, X_3]$ $\rightsquigarrow$ $\mathrm{Crit}(f_n) = \mathrm{Hilb}^n(\mathbb{C}^3)$

- a quiver description: $Q =$



$\Rightarrow \mathbb{C}Q \cong \mathbb{C}\langle X_1, X_2, X_3 \rangle$

Let $W = a_1a_2a_3 - a_2a_1a_3$ $\Rightarrow \partial_{a_i} W = [a_{i+1}, a_{i+2}]$

$J(Q,W) = \mathbb{C}[x_1, x_2, x_3]$

For dimension vector $\underline{v} = n$, the framed moduli space of Q $\cong \mathrm{Hilb}_0^n(\mathbb{C}^3)$