# Dimension in the Langlands programme

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## 1. The Langlands programme

Let $F, k$ be fields.

Conjecture: There is a natural bijection

$$\begin{tikzcd} \left\{\begin{array}{c} n\text{-dimensional representations of} \\ \mathrm{Gal}(\bar{F}/F) \text{ over } k \end{array}\right\} \arrow[r] & \left\{\begin{array}{c} \text{admissible representations} \\ \text{of } SL_n(F) \\ \text{over } k \end{array}\right\} \arrow[d, "\mathrm{Cdim}"] \\ & \mathbb{Z}_{\geq 0} \end{tikzcd}$$

Canonical dimension

+ adjectives!

Examples

- $F = \mathbb{Q}$, $n = 2$ : Fermat's Last Theorem.
- $n = 1$, char $k = 0$ : class field theory

From now on, let char $k = p$, $F$ a finite extension of the $p$-adic numbers $\mathbb{Q}_p$.

Fact: Every (admissible) finite-dimensional rep of $SL_n(F)$ (over $k$) is trivial.

Theorem ($T_0$): Every admissible rep $V$ of $SL_n(F)$ ($\neq SL_2(\mathbb{Q}_p)$) has $\mathrm{Cdim}(V) \neq 1$.

## 2. Canonical dimension

Proposition: Let $G$ $(=SL_n(F))$ be a p-adic Lie group. Then

- $G$ has a compact open subgroup $H$, $(=SL_n(\mathcal{O}_F))$ such that
- $H \geq H^p \geq H^{p^2} \geq \cdots$ is an open neighbourhood basis of the identity.

$G = (\mathbb{Q}_p, +)$, $H = (\mathbb{Z}_p, +)$ $H^{p^n} = p^n \mathbb{Z}_p$

Def: $V$ is admissible if $V = \bigcup_{n \geq 0} V^{H^{p^n}}$, where each $V^U = \{v \in V \mid x \cdot v = v \ \forall x \in U\}$ is finite-dimensional over $k$.

GROWTH

Prop/Def: As $n \to \infty$, $\dim_k(V^{H^{p^n}}) \sim p^n \operatorname{Cdim}(V^*)$

In fact, $\operatorname{Cdim}: R\text{-mod} \longrightarrow \mathbb{Z}_{\geq 0}$

for Auslander-Gorenstein rings $R$.

Lemma: The Krull dimension

$$K(R) \leq |\{\operatorname{Cdim}(M) \mid M \in R\text{-mod}\}| - 1$$

Lemma: A rep $V$ of $G$ is admissible $\iff$ $V^*$ is a finitely-generated module over the Iwasawa algebra

$$kH = \varprojlim_{n \geq 0} k[H/H^{p^n}]$$

Def : $\mathrm{Cdim}(M) = \dim H - \min\{ j \mid \mathrm{Ext}^j_{kH}(M, kH) \neq 0 \}$

COHOMOLOGY

GEOMETRY :

Prop : $kH$ has a filtration s.t. $\mathrm{gr}\, kH \cong k[X_1, \ldots, X_d]$ ($d = \dim H$), and

$$\mathrm{Cdim}(M) = \dim V(\mathrm{Ann}(\mathrm{gr}\, M))^{\subseteq \mathbb{A}^d}.$$