Group Cohomology 30th Oct.

Definition of $H_* G$

Recap: • If M is a G-module, the group of coinvariants of M is
$$M_G = \frac{M}{\langle gm - m \rangle} \quad \text{(i.e. quotient by } G\text{-action)},$$

• The functor $(-)_G$ is exact.

• The homology of G is
$$H_i(G) = H_i(F_G)$$
where $\varepsilon : F \to \mathbb{Z}$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

$H_i(F_G)$ is independent of the choice of resolution, so choose the standard one. Write $C_*(G)$ for the chain complex in this case. This means:

We have $(n+1)$-tuples $(g_0, \dots, g_n)$. Let
$$(g_0, \dots, g_n) \sim (gg_0, \dots, gg_n) \;\; \forall g \in G$$
giving equiv. classes $[g_0, \dots, g_n]$.

$C_n(G)$ has basis consisting of these classes, and $\partial : C_n(G) \to C_{n-1}(G)$ is $\partial = \sum_{i=0}^{n} (-1)^i d_i$ where
$$d_i[g_0, \dots, g_n] = [g_0, \dots, \hat{g}_i, \dots, g_n].$$

$C_*(G)$ is called the homogeneous chain complex of G.

There is also a non-homogeneous description using bar notation:

The $\mathbb{Z}$-basis is n-tuples $[g_1 | \cdots | g_n]$, and $\partial = \sum_{i=0}^{n} (-1)^i d_i$ where
$$d_i [g_1 | \cdots | g_n] = \begin{cases} [g_2 | \cdots | g_n] & i = 0 \\ [g_1 | \cdots | g_i g_{i+1} | \cdots | g_n] & 0 < i < n \\ [g_1 | \cdots | g_{n-1}] & i = n \end{cases}$$

Note: this is a slight abuse of notation: $[g_1 | \cdots | g_n]$ now denotes an equivalence class where before it was just a single basis element.

In low dimensions, $C_*(G)$ has the form
$$C_2(G) \xrightarrow{\partial} C_1(G) \xrightarrow{0} \mathbb{Z} \quad \text{where}$$
$\partial [g | h] = [h] - [gh] + [g]$. So $H_0 G = \mathbb{Z}$ and $H_1 G = G^{ab}$.

4. Topological Interpretation

From Lecture 2: If Y is a $K(G,1)$-complex with universal cover X, then $C_*(X)$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

From Lecture 3: $C_*(Y) \cong C_*(X)_G$.

Combining these, and taking homology,

Prop. 4.1

If Y is a $K(G,1)$-complex then $H_* G \cong H_*(Y)$.

Note: sometimes this is taken to be the definition of $H_* G$.

Example

Let Y be a boquet of $|S|$ circles (where S is some set).

Since Y is a $K(F(S),1)$,

$$H_i(F(S)) = H_i(Y) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}S = F(S)^{ab} & i=1 \\ 0 & i>1 \end{cases}$$

Example

Let $g \in \mathbb{Z}_{>0}$, and $G = \langle a_1, \dots, a_g, b_1, \dots, b_g \mid \prod_{i=1}^{g} [a_i, b_i] \rangle$ (the fundamental group of a closed orientable surface Y of genus g). Then Y is a $K(G,1)$:

The universal cover of Y is non-cpct. since G is infinite. Hence by manifold theory, $H_i X = 0$ for $i > 0$.

Alternatively, $\tilde{Y}$ is the hyperbolic plane tiled by $4g$-gons so is contractible and hence Y is a $K(G,1)$. So

$$H_i G = H_i Y = \begin{cases} \mathbb{Z} & i=0,2 \\ \mathbb{Z}^{2g} & i=1 \\ 0 & i>2 \end{cases}.$$

Example

Let $G = \langle S \mid r \rangle$ be any 1-relator group, and Y be its presentation complex

$$\bigvee_{s \in S} S^1_s \cup_r e^2 \quad \leftarrow \text{2-cell}$$

So $\pi_1 Y = G$.

Thm: • If r is not a power u^n in $F(S)$ then Y is a $K(G,1)$

• If r is a power, then G has torsion so there doesn't exist a finite-dimensional $K(G,1)$.

In the first case:

It can be shown that $C_* Y$ has the form

$$\mathbb{Z} \xrightarrow{\partial} \mathbb{Z}S \xrightarrow{0} \mathbb{Z}$$

where $\partial(1)$ is the image of r in $\mathbb{Z}S = F(S)^{ab}$. So

$$H_0 G = \mathbb{Z}$$

$$H_1 G = G^{ab}$$

$$H_2 G = \begin{cases} \mathbb{Z} & r \in [F(S), F(S)] \\ 0 & \text{o/w} \end{cases}$$

$$H_i G = 0 \text{ for } i > 2.$$

5. Hopf's Theorems

We will need:

Hurewicz Theorem

If $\pi_i X = 0$ for $i < n$ $(n \geq 2)$ then $H_i X = 0$ for $0 < i < n$, and the Hurewicz map $h : \pi_n X \to H_n X$ is an isomorphism.

Lemma 5.1

Let $F_n \xrightarrow{d_n} \dots \to F_0 \to \mathbb{Z} \to 0$ be an exact sequence of projective $\mathbb{Z}G$-modules. Then $H_i G \cong H_i(F_G)$ for $i < n$ and there is an exact sequence

$$0 \to H_{n+1}(G) \to (H_n F)_G \to H_n(F_G) \to H_n(G) \to 0.$$

Proof

Extend F to a full resolution F^+ by choosing a projective module mapping onto $\ker d_n$: ~~(typo in book here)~~

$$\begin{tikzcd} \cdots \arrow[r] & F'_{n+2} \arrow[r] & F'_{n+1} \arrow[r] \arrow[d, two heads] & F_n \arrow[r, "d_n"] & F_{n-1} \arrow[r] & \cdots \arrow[r] & F_0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \\ & & \ker d_n := H_n F \arrow[ur] & & & & & & \end{tikzcd}$$

by projectivity

It can be checked that $H_i(F_G^+) = H_i(F_G)$ for $i<n$ and that there is an exact sequence

$$0 \to H_{n+1}(F_G^+) \to A \to H_n(F_G) \to H_n(F_G^+) \to 0$$

where $A = \operatorname{coker}\{(F'_{n+2})_G \to (F'_{n+1})_G\}$.

By right exactness of $(-)_G$, we have $A = (H_n F)_G$

Thm. 5.2

For a connected CW-complex Y there is a canonical map

$$\psi : H_* Y \to H_* \pi \quad \text{where } \pi = \pi_1 Y.$$

If $\pi_i Y = 0$ for $1 < i < n$ (for some $n \geq 2$) then ψ is an isomorphism for $i<n$, and

$$\pi_n Y \xrightarrow{h} H_n Y \xrightarrow{\psi} H_n \pi \to 0 \quad \text{is exact.}$$

Proof

Let X be the universal cover of Y and F a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\pi$. Then $C_*(X)$ is a complex of free $\mathbb{Z}\pi$-modules augmented over $\mathbb{Z}$. By the Fundamental Theorem (1.7.4), we get a chain map over $\mathbb{Z}\pi$

$$C_*(X) \to F$$

which is well-defined up to homotopy.

Applying $(-)_\pi$, we get a map $C_*(Y) \to F_\pi$ which induces the map $\psi : H_* Y \to H_* \pi$.

Fact: $\pi_i X \cong \pi_i Y$ for $i > 1$.

So $\pi_i X = 0$ for $i < n$ (by our assumption)

By the Hurewicz theorem, we know that

$$h : \pi_n X \longrightarrow H_n X$$

is an isomorphism, so our complex becomes a partial free resolution

$$C_n(X) \longrightarrow \cdots \longrightarrow C_0(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

where the n^{th} homology is the group $Z_n X$ of n-cycles of X. (Same as kernel of first map).

By the lemma, $H_i Y \cong H_i \pi$ for $i < n$, and there is an exact sequence

$$Z_n X \longrightarrow Z_n Y \xrightarrow{\tilde{\psi}} H_n \pi \rightarrow 0.$$

where $\tilde{\psi}$ is the composition

$$Z_n Y \rightarrow H_n Y \xrightarrow{\psi} H_n \pi.$$

So $H_n X \rightarrow H_n Y \xrightarrow{\psi} H_n \pi$ is exact.

We have
$$\begin{tikzcd}
\pi_n X \arrow[r, "h", "\cong"'] \arrow[d, "\cong"'] & H_n X \arrow[d] \\
\pi_n Y \arrow[r, "h"] & H_n Y
\end{tikzcd}$$

which gives an exact sequence

$$\pi_n Y \xrightarrow{h} H_n Y \xrightarrow{\psi} H_n \pi \longrightarrow 0$$

as required. $\blacksquare$

Thm. 5.3

If $G = F/R$ (where F is free) then

$$H_2 G \cong R \cap \frac{[F,F]}{[F,R]}.$$

Proof

Let $F = F(S)$ and Y be a boquet of circles indexed by S (so $F = \pi_1(Y)$). Let $\tilde{Y}$ be the connected regular covering space of Y corresponding to the normal subgroup R. Choose a basepoint $\tilde{v} \in \tilde{Y}$ lying over the vertex of Y. Then $G = F/R$ is the group of deck transformations of $\tilde{Y}$. For $f \in F$, f is a path in Y. Write $\tilde{f}$ for the lift of f starting at $\tilde{v}$. This path ends at $\tilde{f}\tilde{v}$, and $\tilde{f}$ is the image of f in G.

$C_* \tilde{Y}$ is a complex of free $\mathbb{Z}G$-modules, and gives a partial resolution

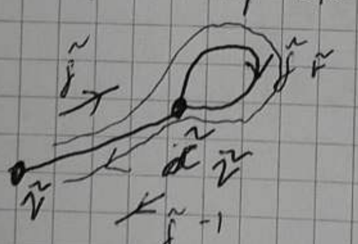
$$C_1\tilde{Y} \to C_0\tilde{Y} \to \mathbb{Z} \to 0.$$

By the lemma, $H_2 G \cong \ker\{(H_1\tilde{Y})_G \to H_1 Y\}$

$H_1\tilde{Y} \cong (\pi_1 \tilde{Y})^{ab} = R^{ab}$

Claim: $H_1\tilde{Y} \cong R^{ab}$ is an isomorphism of G-modules, where the G-action on R^{ab} is induced by the conjugation action of F on R.

Proof: It can be shown that the isomorphism $R^{ab} \xrightarrow{\cong} H_1\tilde{Y}$ is induced by the map $d: R \to H_1\tilde{Y}$ given by:
For $r \in R$, the lift of r, $\tilde{r}$ is a closed path in $\tilde{Y}$. This is a 1-cycle and therefore an element of $H_1\tilde{Y}$.
Define $dr = \tilde{r}$.
To show that d respects the G-action, take a lift of frf^{-1}:



so $d(g r g^{-1}) = \tilde{g}\, dr$

G-action

Hence

$$\begin{tikzcd}
(H_1 \tilde{Y})_G \arrow[r, phantom, "\cong"] \arrow[d, "\text{covering}"'] & (R^{ab})_G \arrow[r, phantom, "="] \arrow[d] & \frac{R}{[F,R]} \arrow[d] \\
H_1 Y \arrow[r, phantom, "\cong"] & F^{ab} \arrow[r, phantom, "="] & \frac{F}{[F,F]}
\end{tikzcd}$$

inclusion of R into F

so $H_2 G \cong \ker\left\{ \frac{R}{[F,R]} \longrightarrow \frac{F}{[F,F]} \right\} = \frac{R \cap [F,F]}{[F,R]}$

By the Lemma

<u>Remarks</u>

- R^{ab} is called the <u>relation module</u> of the presentation.
- This means we can (roughly) think of $H_2 G$ as commutator relations $\prod [a_i, b_i] = 1$ in G, modulo those that hold trivially.

<u>Corollary</u>

If $G = \frac{F(S)}{R}$ then there is an exact sequence of G-modules

$$0 \longrightarrow R^{ab} \xrightarrow{d} \mathbb{Z}G^{(S)} \xrightarrow{\partial} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

path-lifting

where $\mathbb{Z}G^{(S)}$ is free with basis $(e_s)_{s \in S}$, and $\partial e_s = \bar{s} - 1$ (where $\bar{s}$ is the image of s in G).