Recall

(T, η, μ) a monad on $\mathcal{C}$ $\qquad \eta : id_{\mathcal{C}} \to T$

$\qquad\qquad\qquad\qquad\qquad\qquad \mu : T^2 \to T$

$\mathcal{C}_T$ Kleisli category

↳ Obj = obj of $\mathcal{C}$

↳ Morph $X \nrightarrow Y$ in $\mathcal{C}_T$ is morph $X \to TY$ in $\mathcal{C}$

$\mathcal{C}^T$ Eilenberg - Moore category / category of algs.

↳ Obj $(A \in \mathcal{C}, \alpha : TA \to A)$ st

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "id_A"'] & TA \arrow[d, "\alpha"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "\mu_A"] \arrow[d, "T\alpha"'] & TA \arrow[d, "\alpha"] \\ TA \arrow[r, "\alpha"'] & A \end{tikzcd}$$

↳ Morph $(A, \alpha) \to (B, \beta)$ is morph $f : A \to B$ in $\mathcal{C}$ st

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TB \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Adj_T for (T, η, μ) a monad on $\mathcal{C}$

↳ Obj adjunctions $\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{D}$ ($F \dashv U$) w/ $\eta : id_{\mathcal{C}} \to UF$, $\varepsilon : FU \to id_{\mathbb{D}}$

st $T = UF$, $\eta = \eta$, $\mu = U \varepsilon F$

$\qquad F : \mathcal{C} \to \mathbb{D} \qquad\qquad F' : \mathcal{C}' \to \mathbb{D}'$

↳ Morph. $(F, U, \eta, \varepsilon) \to (F', U', \eta', \varepsilon')$ is functor $K : \mathbb{D} \to \mathbb{D}'$ st.

$$\begin{tikzcd} \mathbb{D} \arrow[rr, "K"] \arrow[dr, shift left, "U"] & & \mathbb{D}' \arrow[dl, shift left, "U'"] \\ & \mathcal{C} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F'"] & \end{tikzcd}$$

Prop. 5.2.12 $\mathcal{C}_T$ ($\mathcal{C}^T$) is initial (final) in Adj_T. ie. if $F \dashv U$ induces T then

$$\begin{tikzcd}
\mathcal{C}_T \arrow[r, dashed, "\exists! J"] \arrow[dr, shift left, "U^T"] & \mathcal{D} \arrow[r, dashed, "\exists! K"] \arrow[d, shift left, "U"] & \mathcal{C}^T \arrow[dl, shift left, "U^T"] \\
& \mathcal{C} \arrow[ul, shift left, "F^T"] \arrow[u, shift left, "F"] \arrow[ur, shift left, "F^T"] &
\end{tikzcd}$$

w/ $F^T A = (TA, \mu_A)$ free alg.

$U^T(A, \alpha) = A$ forgetful functor

$F_T A = A \qquad (f : A \to B) \mapsto (\eta_B \circ f : A \nrightarrow B)$

$U_T A = TA \qquad (g : A \nrightarrow B) \mapsto (\mu_B \circ Tg : TA \to TB)$

Def. 5.3.1

- $\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D}$ ($F \dashv U$) monadic if $K : \mathcal{D} \to \mathcal{C}^T$ above is an equivalence
- $U : \mathcal{D} \to \mathcal{C}$ monadic if it admits a left adjoint defining a monadic adjunction

---

5.5 Recognising Categories of Algebras

Thm 5.5.1 (Monadicity Thm)

A right adjoint functor, $U : \mathcal{D} \to \mathcal{C}$, is monadic if & only if it creates coequalisers of U-split pairs

Recall:

Def. 5.4.8. Functor $U : \mathcal{D} \to \mathcal{C}$

- U-split coequaliser is a pair of morphisms, $f, g : x \to y$, in $\mathcal{D}$ st.

$\exists\ h, z, s, t$ st.

$$\begin{tikzcd} Ux \arrow[r, shift left, "Uf"] \arrow[r, shift right, "Ug"'] & Uy \arrow[r, "h"] \arrow[l, bend left=60, "t"] & Z \arrow[l, bend left=60, "s"] \end{tikzcd}$$

is a split coequaliser in $\mathcal{C}$

ie st. $hUf = hUg$, $hs = id_Z$, $Ug\,t = id_y$, $Uf\,t = sh$

- U creates coequalisers of U-split pairs if any U-split coequaliser admits a coequaliser in $\mathcal{D}$ whose image under U is isomorphic to the fork underlying the given U-split coequaliser diagram in $\mathcal{C}$ & if any such fork in $\mathcal{D}$ is a coequaliser.

ie. U is monadic if $F \dashv U$ is an adjunction inducing a monad, T, on $\mathcal{C}$ giving rise to

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "K"] \arrow[dr, shift left, "U"] & & \mathcal{C}^T \arrow[dl, shift left, "U^T"] \\ & \mathcal{C} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F^T"] & \end{tikzcd}$$

then TFAE

- K is an equivalence / iso of cats.
- U creates / strictly creates coequalisers of U-split pairs.

Previously ended w/ corollary 5.5.3 listing some categories whic are monadic over Set.

Now:

Recall:

Def. 3.8.7. A category, J, is filtered if $\exists$ a cone under every finite diagram in J.

A filtered colimit is a colimit where the indexing category is filtered.

Explicitly

1. J is not empty
2. For $j, j' \in J$ $\exists$ $k \in J$ & $f : j \to k$, $f' : j' \to k$ in J
3. For $u, v : i \to j$ $\exists$ $k \in J$ & $w : j \to k$ w/ $wu = wv$

1. $\Rightarrow \exists$ cone under the diagram $\{\ \} \to J$ (terminal obj. diagram)
2. $\Rightarrow \exists$ cone under the diagram $\{j \ \ j'\} \to J$ (coproduct diagram)
3. $\Rightarrow \exists$ cone under the diagram $\{i \rightrightarrows j\} \to J$ (coequaliser diagram)

Nb. nothing about these cones being universal.

Any finite diagram can be built from these pieces so this is equivalent to definition in [Riehl]. Cf. any finite limit is built from products & equalisers.

Def. 5.5.4. A functor is finitary if it preserves filtered colimits.

ie. J filtered & $F : \mathcal{C} \to \mathcal{D}$ a functor, F is finitary if for all diagrams $K : J \to \mathcal{C}$ w/ a colimit cone, (c, λ), the image of this is again a colimit cone over the diagram $FK : J \to \mathcal{D}$ ie $(Fc, F\lambda)$ is a colimit cone

Nb. (c, λ_c) a cone under $c \Rightarrow \lambda_j : Kj \to c$ form a nat. trans. $K \to \Delta_c$
& $(Fc, F\lambda)$ a cone under $c \Rightarrow F\lambda_j : FKj \to Fc$ form a nat. trans. $FK \to \Delta_c$
see def. 3.1.2 $\qquad$ Δ_c: const. functor at c.

Nb. If a right adjoint, U, is finitary so is the corresponding monad, $T = UF$, b/c left adjoints, F, preserve all colimits

Def. 5.5.5 A category, $\mathcal{A}$, is a category of models for an algebraic theory if there is a finitary monadic functor $U : \mathcal{A} \to \mathbf{Set}$.

Such a category is always locally finitely presentable
ie cocomplete & equipped w/ a set of objects, S, st
a). Every object of $\mathcal{C}$ is a colimit of a diagram in S
b) For $s \in S$ the functor $\mathcal{C}(s, -) : \mathcal{C} \to \mathbf{Set}$ preserves filtered colimits

Cor. 5.5.6 $\quad U : \mathbf{Top}^{CH} \to \mathbf{Set}$ is monadic (but not finitary) $\to$ so not algebraic

despite this, $\mathbf{Top}^{CH}$ has many properties of an algebraic theory, such as being cocomplete
$\rightsquigarrow \mathbf{Top}^{CH}$ is as "nice" as topological spaces get

I'm an algebraist, so my definition of nice is probably different to a topologists (who would likely say "boring").

Exm 5.5.7
1. $U : R\text{-Mod} \to \mathbf{Ab}$ is monadic
The induced monad on $\mathbf{Ab}$ is $R \otimes_{\mathbb{Z}} - : \mathbf{Ab} \to \mathbf{Ab}$

ie view R as an abelian group (forget multiplication). Then

$$TA = R \otimes_{\mathbb{Z}} A = \langle (r, a) \in R \times A \mid (r, a) + (r, a') = (r, a + a'),\ (r, a) + (r', a) = (r + r', a) \rangle$$

Note $(nr, a) = (r, na)$ for $n \in \mathbb{Z}$.

2. $U : \mathrm{Ring} \to \mathrm{Ab}$ is monadic & the induced monad on Ab is the free monoid monad, $TA = \bigoplus_{n \geq 0} A^{\otimes n}$.

3. If $\mathcal{C}$ has coproducts (or just copowers $X^{\oplus n}$) & J is small then the functor $\mathcal{C}^J \to \mathcal{C}^{\mathrm{obj}\, J}$ carrying a diagram (functor $J \to \mathcal{C}$) to the $\mathrm{Obj}\, J$ indexed family of objects in this diagram (image of the functor) is monadic.

Special case:

$\mathbb{B}G$ = View the gp G as a 1 object category

$\mathrm{Vect}_K^{\mathbb{B}G} \cong \mathrm{Rep}_K(G)$

$\mathrm{Vect}_K^{\mathbb{B}G}$ obj functors $\mathbb{B}G \to \mathrm{Vect}_K$. These are exactly G-reps

Let $*$ be the single object of $\mathbb{B}G$. Then $F* = V$ is a K-vector space. If $g : * \to *$ is an element of G then $Fg : V \to V$ is a linear map ie $Fg \in \mathrm{End}\, V$. Let $\rho(g) = Fg$. Then (V, ρ) is a G-rep.

Pf. $\rho(gh) = F(gh) = (Fg)(Fh) = \rho(g)\rho(h)$

$\rho(e) = F(id_*) = id_{F*} = id_V$

Further, let $F, F' : \mathbb{B}G \to \mathrm{Vect}_K$ be functors & let $\theta : F \to F'$ be a nat. trans. Then $\forall\ g : * \to *$ we have

$$\begin{tikzcd} F* \arrow[r, "\theta_*"] \arrow[d, "Fg"'] \arrow[dr, phantom, "\circlearrowright"] & F'* \arrow[d, "F'g"] \\ F* \arrow[r, "\theta_*"'] & F'* \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} V \arrow[r, "\Theta"] \arrow[d, "\rho(g)"'] & V' \arrow[d, "\rho'(g)"] \\ V \arrow[r, "\Theta"'] & V' \end{tikzcd}$$

↗ just a new name for θ_*

$\Rightarrow \Theta : V \to V'$ is an equivariant map ie a morphism of reps.

The functor $\mathrm{Vect}_k^{BG} \longrightarrow \mathrm{Vect}_k^{\mathrm{Obj}\, BG}$ is monadic

$\mathrm{Obj}\, BG = \{*\}$ so $\mathrm{Vect}_k^{\mathrm{Obj}\, BG}$ is exactly single objects of Vect_k ie just vector spaces.

ie $\mathrm{Vect}_k^{BG} \to \mathrm{Vect}_k^{\mathrm{Obj}\, G}$ is the functor $\mathrm{Rep}(G) \longrightarrow \mathrm{Vect}_k$

$(V, \rho) \longmapsto V$

sending a G-rep to the representing space.

4. $U : \mathrm{Cat} \longrightarrow \mathrm{DirGraph}$ is monadic.

The induced monad on DirGraph sends a directed graph (V, E) to the corresponding graph w/ the same vertices & adding

- a loop "identity" edge $v \circlearrowleft$ for all $v \in V$
- a "composite" edge $v \longrightarrow w$ for all $v, w \in V$ when there exist vertices & edges $v \to v' \to v'' \to \cdots \to w$

Nb. can replace DirGraph w/ rDirGraph → reflexive directed graphs

ie directed graphs (V, E) where $\exists$ an edge $v \circlearrowleft$ for all $v \in V$.

There are various monadicity theorems w/ different conditions.

These are termed (Blank) "tripleability" theorems

↳ old term for monad

The result we stated is the precise tripleability theorem (PTT).

Another variant is the reflexive tripleability theorem (RTT)

Prop 5.5.8 [RTT]

If $U : \mathbb{D} \to \mathcal{C}$ has a left adjoint &

1. $\mathbb{D}$ has coequalisers of reflexive pairs
2. U preserves coequalisers of reflexive pairs

$f, g : A \to B$ a reflexive pair

is $\exists$ common section $s : B \to A$

st $f \circ s = g \circ s = id_B$

3. U reflects isomorphisms

Then U is monadic.

Thm 5.5.9: The contravariant power set functor, $P : \mathbf{Set}^{op} \to \mathbf{Set}$, is monadic.

Overkill: Nice that it holds in an arbitrary topos.

Lem 5.5.10. For a pullback diagram of monomorphisms

$$\begin{tikzcd} A' \arrow[r, "g"] \arrow[d, tail, "a"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B' \arrow[d, tail, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

the square

$$\begin{tikzcd} PB' \arrow[r, "g^{-1}"] \arrow[d, "b_*"'] & PA' \arrow[d, "a_*"] \\ PB \arrow[r, "f^{-1}"'] & PA \end{tikzcd}$$

commutes.

Nb. f^{-1} is the preimage map

$f^{-1}(Y) = \{ x \in A \mid f(x) \in Y \} \in PA$

↳ $Y \in PB$

Nb. a_* is given by

$a_*(X) = \{ a(x) \mid x \in X \} \in PA$

↳ $X \in PA'$

Pf. for $X \in PB'$ the rectangle

$$\begin{tikzcd} Y \arrow[r, "g|_Y"] \arrow[d, hook] & X \arrow[d, hook] \\ A' \arrow[r, "g"] \arrow[d, tail, "a"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B' \arrow[d, tail, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes. (b/c the top square is just inclusion & restriction & the bottom square commutes by assumption.)

Then
$$a_* g^{-1}(X) = a_*\{x \in A' \mid g(x) \in X\}$$
$$= \{a(x) \mid x \in A' \text{ w/ } g(x) \in X\}$$
This is Y precisely if the top square is a pullback.

Similarly, $f^{-1} b_*(X) = Y$ if the rectangle is a pullback.

Ex. 3.1.VIII says these conditions are equivalent. $\square$

Ex 4.5.V any mono $f: A \rightarrowtail B$ def's a pullback

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, tail, "f"] \\
A \arrow[r, tail, "f"'] & B
\end{tikzcd}$$

& so a special case of this lemma is that

$$\begin{tikzcd}
PA \arrow[r, "f_*"] \arrow[rr, bend right, "id_{PA}"'] & PB \arrow[r, "f^{-1}"] & PA
\end{tikzcd}$$

$$\left.\begin{aligned}
f^{-1} f_*(\underset{A}{X}) &= f^{-1}\{f(x) \mid x \in X\} \\
&= \{b \in B \mid f(b) \in \{f(x) \mid x \in X\}\} \\
&= \{b \in B \mid b \in X\} \\
&= X
\end{aligned}\right]$$

Direct calculation showing same thing as above.

Pf. Thm 5.5.9.

$$\begin{aligned}
\mathbf{Set}(A, PB) &\cong \mathbf{Set}(A, \mathbf{Set}(B, \Omega)) && \text{b/c } \Omega \text{ represents } P \\
&\cong \mathbf{Set}(A \times B, \Omega) && \text{tensor/hom} \\
&\cong \mathbf{Set}(B \times A, \Omega) && \mathbf{Set} \text{ sym. mon. cat} \\
&\cong \mathbf{Set}(B, \mathbf{Set}(A, \Omega)) && \text{tensor/hom} \\
&\cong \mathbf{Set}(B, PA) && \text{b/c } \Omega \text{ represents } P
\end{aligned}$$

aka Currying

So, P is its own left adjoint

Set has finite limits, including equalisers of coreflexive pairs.

$\therefore$ dually Set has coequalisers of reflexive pairs

↳ $f, g : A \to B$ have a common retract, $r : B \to A$, st $r \circ f = r \circ g = id_A$

Prp 5.5.8 part 1 ✓

Let $f, g : A \to B$ be a coreflexive pair. ie $\exists\ r : B \to A$ st.

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[rr, bend right, "id_A"'] & B \arrow[r, "r"] & A
\end{tikzcd}$$

commutes. Let E be an equaliser of this pair, ie st.

$$\begin{tikzcd}
E \arrow[r, tail, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}$$

commutes. Then the square

$$\begin{tikzcd}
E \arrow[r, tail, "e"] \arrow[d, tail, "e"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, tail, "f"] \\
A \arrow[r, tail, "g"'] & B
\end{tikzcd}$$

is a pullback of monos: the existence of r implies f & g are split mono & hence the legs of any cone over this diagram must be equal

$$\begin{tikzcd} X \arrow[rrd, bend left, "p"] \arrow[rdd, bend right, "q"'] \arrow[rrdd, "k"] & & \\ & E \arrow[r, tail, "e"] \arrow[d, tail, "e"'] & A \arrow[d, tail, "f"] \\ & A \arrow[r, tail, "g"'] & B \end{tikzcd}$$

$k = fp = gq$

$\Rightarrow rk = rfp = rgq$

$\Rightarrow rk = p = q$

So $p = q$. $\exists$ $h : X \to E$ by defining $h(x)$ = the unique element of E which maps to $p(x)$ under e.

This h is unique in making the pullback diagram w/ cone commute

Using $f^{-1} f_* = id_{PA}$ we have that

$$\begin{tikzcd} PB \arrow[r, shift left, "f^{-1}"] \arrow[r, shift right, "g^{-1}"'] & PA \arrow[l, bend left=60, "g_*"] \arrow[r, "e^{-1}"] & PE \arrow[l, bend left=60, "e_*"] \end{tikzcd}$$

is a split coequaliser Prp 5.5.8 part 2 ✓

Note P is faithful: given $f, g : A \to B$ construct composites

$$\begin{tikzcd} B \arrow[r, "\eta"] & PB \arrow[r, shift left, "f^{-1}"] \arrow[r, shift right, "g^{-1}"'] & PA \end{tikzcd}$$

$\eta(b) = \{b\}$ singleton map.

These transpose to define maps $A \times B \to \Omega$ classifying the subsets

$(1, f) : A \rightarrowtail A \times B$

$(1, g) : A \rightarrowtail A \times B$

By the universal property of the subobject classifier if $f^{-1} = g^{-1}$ then $f = g$

Ex. 1.6.III faithful functors reflect monos & epis.

In Set (or any elementary topos) iso $\iff$ mono & epi.

$\therefore$ P reflects isos   Prp 5.5.8 part 3 ✓

$\square$

5.6 Limits & Colimits in Categories of Algebras

Definition of a monadic functor is <u>very</u> technical.

Goal: Use categorical universal algebra to describe common features of categories equivalent to a category of algebras.

These results apply to

Set_*, Mon, Grp, Ab, Ring, CRing, R-Mod, $\mathrm{Aff}_{\mathbb{K}}$, Set^{BG} (G-sets), Lattice & $\mathrm{Top}^{\mathrm{CH}}$

These are monadic over Set. $\leadsto$ can look at categories monadic over some other category too.

These results generally <u>aren't</u> true for Poset, Top & Field which aren't monadic over Set.

↳ U: Field $\to$ Set has no adjoint (Ex 4.1.12 $\exists$ functions $\mathbb{Z} \to K$ for K of any char but only field homs $K \to \mathbb{Z}$ for K char. 0)

Lem 5.6.1

If $U: \mathcal{A} \to \mathcal{C}$ is monadic then U reflects isos, ie $\forall$ morphisms $f: a \to a'$ in $\mathcal{A}$ if Uf is an iso in $\mathcal{C}$ then f is an iso in $\mathcal{A}$.

Call functor **conservative** if it reflects isos.

Pf. Sufficient to show $U^T: \mathcal{C}^T \to \mathcal{C}$ has this property

Morph $(A, \alpha) \xrightarrow{f} (A', \alpha')$ in $\mathcal{C}^T$ is a morph $f: A \to A'$ in $\mathcal{C}$ st.

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & TA' \arrow[d, "\alpha'"] \\ A \arrow[r, "f"'] & A' \end{tikzcd}$$

When f^{-1} exists it follows that

$$\begin{tikzcd} TA' \arrow[r, "Tf^{-1}"] \arrow[d, "\alpha'"'] \arrow[dr, phantom, "\circlearrowleft"] & TA \arrow[d, "\alpha"] \\ A' \arrow[r, "f^{-1}"'] & A \end{tikzcd}$$

& so f^{-1} existing in $\mathcal{C}$ means the inverse $(A', \alpha') \xrightarrow{F^{-1}} (A, \alpha)$ exists.
Then $U^T F = f$ & $U^T F^{-1} = f^{-1}$ so U^T reflects isos. $\square$

Cor. 5.6.2 Any bij. cts. map between compact Hausdorff spaces is a homeomorphism

Pf. $U : \mathrm{Top}^{CH} \to \mathrm{Set}$ sends a bij. cts fct. to a bij. fct. & ∴ iso & ∴ original map is an iso in Top^{CH}. $\square$

[As it should be, bijections not being isos is a "bad" feature of Top but not Top^{CH}].

Cor 5.6.4 Any bij. hom in a category of models for an algebraic theory is an isomorphism.

The contrapositive of this shows Poset & Top <u>aren't</u> monadic over Set, they have bijective homs which aren't invertible.

<u>Limits & Colimits</u>

$K : J \to \mathcal{C}$ diagram, $U : \mathcal{A} \to \mathcal{C}$ creates limits of shape K if whenever $\lim_{\mathcal{C}} K$ exists $\lim_{\mathcal{A}} K$ exists & further $\lim_{\mathcal{C}} K = U(\lim_{\mathcal{A}} K)$

Thm 6.5.6

A monadic functor, $U : \mathcal{A} \to \mathcal{C}$, creates

i) all limits $\mathcal{C}$ has

ii) all colimits $\mathcal{C}$ has which are preserved by T & T^2

Cor. 5.6.6.

the inclusion functor has a left adjoint

The inclusion of a reflective subcategory creates all limits. In particular, a reflective subcategory of a complete category is complete.

Pf. Such an inclusion is monadic (Prp. 5.3.3)

Cor. 5.6.7 Any category which is monadic over Set is complete, w/ limits created by the forgetful functor.

This is why eg. the product of groups, $G \times H$, has the Cartesian product $G \times H$ as its underlying set, it's "created" by the forgetful functor $U: \mathrm{Grp} \to \mathrm{Set}$ ie. the group product $G \times H$ must just be placing an appropriate group structure on the Cartesian product.

Note U doesn't create colimits hence why in Set $G + H = G \sqcup H$ & in Grp $G + H = G * H$ & $U(G * H) \neq G \sqcup H$

finite if G & H finite

∞ for $G, H \neq \{e\}$

eg. 5.6.8 p-adics $\quad p \in \mathbb{Z}$, p prime $\quad \omega = 0 \to 1 \to 2 \to 3 \to \cdots$

$\omega^{op} = \cdots \to 3 \to 2 \to 1 \to 0$

Not $\mathbb{Z}/p\mathbb{Z}$!

$$\mathbb{Z}_p = \lim_n \mathbb{Z}/p^n\mathbb{Z}$$

ω^{op} diagram, can also def as colim of ω diagram

Limit of $\quad \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z} \twoheadrightarrow \cdots \twoheadrightarrow \mathbb{Z}/p^2\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$ (*)

in the category of rings

$$\mathbb{Z}/p^n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^{n-1}\mathbb{Z}$$
$$m \longmapsto m \bmod p^{n-1}$$

This definition is scary & hard to work w/

Fortunately, $\mathcal{U}: \text{Ring} \to \text{Set}$ creates limits.

$\therefore$ $\mathbb{Z}_p$ is just given by placing a ring structure on the limit of $(*)$ in the category of Sets.

As a set

Have general formula for limits in Set see Eg. 3.2.10

$$\lim_n \mathbb{Z}/p^n\mathbb{Z} = \{(a_1, a_2, \dots) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z} \times \cdots \mid a_n \equiv a_{n+1} \bmod p^n\}$$

specifically inverse limit

Intuition: Colimit identifies sequences which are "eventually equal"

Since $\mathcal{U}$ creates the limit cone it must be that $\mathbb{Z}_p \xrightarrow{proj} \mathbb{Z}/p^n\mathbb{Z}$ is a ring hom $\Rightarrow$ operations have to be def'd componentwise b/c must work seperately for each component.

Cor. 5.6.9. Set is cocomplete

Pf. $P: \text{Set}^{op} \to \text{Set}$ is monadic $\Rightarrow$ Set^{op} is complete

$\Rightarrow$ Set is cocomplete

Colimits in Set created from limit of some diagram in powersets.

Cor. 5.6.10

$\mathcal{U}: R\text{-Mod} \to \text{Ab}$ creates all limits Ab admits.

Pf. $R \otimes_{\mathbb{Z}} - : \text{Ab} \to \text{Ab}$ a monad & has right adjoint $\text{Hom}_{\mathbb{Z}}(R, -)$.

$R \otimes_{\mathbb{Z}} -$ preserves all colimits $\therefore$ Thm 5.6.5 ii) applies to all diagrams in Ab

$\hookrightarrow$ $\mathcal{U}$ creates colimits preserved by T & T^2

Ab & any model for an algebraic theory is cocomplete.
Colimits not preserved by T are harder.
cf. $G * H$ being messed up.

Can def. $G * H$ as "words in $G \sqcup H$ /(relations of G&H)"
Can also def. in terms of free ⊣ forget

$$\begin{tikzcd} \mathrm{Set} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Grp} \arrow[l, shift left, "U"] \end{tikzcd}$$

(This def generalizes to coprod relative to any monadic adjunction)

1st approximation: $G * H$ given by $F(UG \sqcup UH)$ = words in $UG \sqcup UH$
Problem: forgets about relations in G or in H

ie $g h h' g'$ in $F(UG \sqcup UH)$. in $G * H$ it reduces to $g h'' g'$ w/ $h'' = hh'$.

Need to impose relations of G & H. These are generated by words in G & words in H resp.
These generators are elements of $\underbrace{UFUG}_{\substack{\text{set of words} \\ \text{in elements of} \\ G}} \sqcup \underbrace{UFUH}_{\substack{\text{set of words} \\ \text{in elements of } H}}$

Then $F(UFUG \sqcup UFUH)$ is the group of words of words of G or H so each subword is only in G or only in H.
To get $G * H$ we need to impose relations allowing us to evaluate these subwords using the relations of G or H resp.

G & H are groups, which as algebras of a monad is encoded by the corresponding components of the counit

$$\varepsilon_G : FUG \to G \quad \& \quad \varepsilon_H : FUH \to H$$

These take a word in elements of G (resp. H) & evaluate it to get a single element of G (resp. H).

Look at diagram.

evaluate subword in G or H to get resulting word of alternating letters from G & H.

$$\begin{tikzcd}
F(UFUG \sqcup UFUH) \arrow[rr, "F(U\varepsilon_G \sqcup U\varepsilon_H)"] \arrow[dr, "FK"'] & & F(UG \sqcup UH) \\
& FUF(UG \sqcup UH) \arrow[ur, "\varepsilon_{F(UG \sqcup UH)}"'] &
\end{tikzcd}$$

concatenate words of words to a single word.

Ex. 3.3.i.

$K : (UF-) \sqcup (UF-) \Rightarrow UF(- \sqcup -)$ "pulls functor out of colim"

Then $G * H$ is the coequaliser

$$\begin{tikzcd}
\underbrace{F(UFUG \sqcup UFUH)}_{=X} \arrow[r, shift left, "F(U\varepsilon_G \sqcup U\varepsilon_H) = f"] \arrow[r, shift right, "\varepsilon_{F(UG \sqcup UH)} \circ FK = g"'] & \underbrace{F(UG \sqcup UH)}_{=Y} \arrow[r, two heads] & G * H
\end{tikzcd}$$

ie $G * H$ is a quotient of $F(UG \sqcup UH)$ by the relations which make these two maps the same after projecting onto $G * H$

[ie $G * H = Y / ncl_Y(\{ f(x) g(x)^{-1} \mid x \in X \})$

normal closure = intersection of all normal subgps. of Y containing this set

This quotient sets $f(x) g(x)^{-1} = e \Rightarrow f(x) = g(x)$ in quotient ie in $G * H$.]

This generalises:

Prop. 5.6.11

$\mathcal{C}$ cocomplete

$U : \mathcal{A} \to \mathcal{C}$ monadic

TFAE:

- $\mathcal{A}$ cocomplete
- $\mathcal{A}$ has coequalisers

Pf. $i \Rightarrow ii$

cocomplete $\Rightarrow$ all (small) colimits including coequaliser

$ii \Rightarrow i$

can replace $\mathcal{A}$ w/ $\mathcal{C}^T$

as above coequaliser gives all coprods.

Thm 3.4.12 all coprods & coequalisers $\Rightarrow$ cocomplete.

Sufficient properties for a category of algebras to have coequalisers?

& hence be cocomplete

Thm 5.6.12

$T : \mathcal{C} \to \mathcal{C}$ finitary monad

$\mathcal{C}$ complete & cocomplete

$\mathcal{C}$ locally small

Then $\mathcal{C}^T$ is complete & cocomplete.

Proof is 4 pages

Cor. 5.6.14: Any category of models for an algebraic theory is cocomplete