Lecture ν 4

§5 Recognizing Categories of Algebras

Recall:

Def 3.8.7 $\mathcal{I}$ is <u>filtered</u> if it admits a cone under every finite diagram.

A filtered colimit is a colimit of a functor from a filtered category

Explicitly:

- J is not empty
- $\forall j, j' \in J$ $\exists k \in J$ and $f: j \to k$
  $f': j' \to k$
- $\forall u, v: j \to j'$ in J
  $\exists k \in J$ and $w: j' \to k$, s.t.

Def 554 A functor is finitary if it preserves all filtered colimits

If $U : \mathcal{A} \to \mathcal{C}$ is finitary and right adjoin then the corresponding monad is also finitary (F is left adjoint $\Rightarrow$ $\Rightarrow$ preserves colimits)

Def 55.5: A category $\mathcal{A}$ is a category of models for an algebraic theory if there is a finitary monadic functor $U : \mathcal{A} \longrightarrow \mathrm{Set}$

Such a category is always locally presentable

- cocomplete
- $\exists$ a set S of obj. .

a) $\forall a \in \mathcal{A}$ is a colimit of a diagram in S

b) $\forall s \in S$

$\mathcal{A}(s,-) : \mathcal{A} \to Set$ preserves filtered colimits

<u>Ex</u> Top^{CH} – topological spaces, compact and Haus

$U : Top^{CH} \to Set$ is monadic but finitary.

<u>Ex</u> 55.7 :

1) $U: R\text{-Mod} \to Ab$ is monadic

$$T = R \otimes_{\mathbb{Z}} - \quad Ab \to Ab$$

(R-Mod is "algebraic over abelian groups")

2) $U: Ring \to Ab$

T = free monoid monad

$$TA = \bigoplus_{n \geq 0} A^{\otimes n}$$

3) If $\mathcal{C}$ has coproducts and J is small, then

$\mathcal{C}^J \longrightarrow \mathcal{C}^{\mathrm{obj}\, J}$ is monadic

$\mathcal{C}^J$ = Functors $J \longrightarrow \mathcal{C}$

$\mathcal{C}^{\mathrm{obj}\, J}$ = function of objects in $\mathcal{C}$ indexed by obj J.

Special case:

$\mathrm{Vect}_k^{BG} \longrightarrow \mathrm{Vect}_k^{\{*\}} \cong \mathrm{Vect}_k$

(BG group as a category, a single object)

$$\mathrm{Rep}_k(G) \simeq \mathrm{Vect}_k^{BG}$$

This carries a representation to an underlying vect. space.

4.) $U: \mathrm{Cat} \longrightarrow \mathrm{Dir.\,Graph}$ is monadic. The induced monad

$$T: \mathrm{Dir\,Graph} \to \mathrm{Dir.\,Graph}$$

$$(V, E) \longmapsto (V', E')$$

$$V = V'$$

$E' = \{$paths between vertices$\}$

„free category on the digraph (V,E)"

Various monadicity theorems / tripleability theorems

precise tripleability theorem (PTT)

$\parallel$

Beck's monadicity theorem.

Prop: 5.5.8:
Reflexive TT (RTT):

$U: \mathcal{D} \to \mathcal{C}$ has a left adjoint

1) $\mathcal{D}$ has coeq.s of reflexive pairs

2) U preserves coeq.s of reflexive pairs

3) U reflects isom-s.

Then U is monadic

(Reflexive pairs:

A pair $f, g: x \rightrightarrows y$, st it admits a common section $s: y \to x$, $fs = gs = \mathrm{id}_y$)

<u>Ex</u>: $\mathcal{P}: \mathrm{Set}^{op} \to \mathrm{Set}$ is monadic — by RTT.

Lemma 5.5.10 $f: A \to B$ — monomorphism

$$\begin{tikzcd} PA \arrow[r, "f_*"] \arrow[rr, bend right, "\mathrm{id}_{PA}"'] & PB \arrow[r, "f^{-1}"] & PA \end{tikzcd}$$

$\forall X \subseteq A$

$f^{-1} f_*(X) =$

$= f^{-1}(\{ f(x) \mid x \in X \}) =$

$= \{ a \in A \mid \exists x \in \{ f(x) \mid x \in X \},$
s.t. $f(a) = x \} =$

$= \{ a \in A \mid f(a) \in \{ f(x) \mid x \in X \}\}$

$= X$

Thm: 5.5.9

1) $\mathrm{Set}(A, PB) \cong$

$\mathrm{Set}(A, \mathrm{Set}(B, \Omega)) \cong$ Ten-Hom

$\Omega = \{0,1\}$

$\cong \mathrm{Set}(A \times B, \Omega) \cong$

$\cong \mathrm{Set}(B \times A, \Omega) \cong$

$\cong \mathrm{Set}(B, PA) \cong$

$\cong \mathrm{Set}^{op}(PA, B)$

$\Rightarrow$ P is its own left adjoint

2) Set has finite limit (since it is complete) $\Rightarrow$

$\Rightarrow$ has equalizers of coreflexive pairs

$\Rightarrow$ Set^{op} has

coequalizers of reflexive pairs $\Rightarrow$ p1 ✓

3) Let $f, g : A \rightrightarrows B$

be a coreflexive pair

$(\exists r : B \to A,\ r \circ f = r \circ g = id_A)$

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[rr, bend right, "id_A"'] & B \arrow[r, "r"] & A \end{tikzcd}$$

let E be a eq.:

$$\begin{tikzcd} E \arrow[r, tail, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Then the square

$$\begin{tikzcd}
E \arrow[r, tail, "e"] \arrow[d, tail, "e"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}$$

is a pullback of monomorphisms

$(\exists e - \text{retract}) \Rightarrow$

$\Rightarrow f^{-1} f_* = id_{PA} \Rightarrow$

$$\begin{tikzcd}
PB \arrow[r, shift left, "f^{-1}"] \arrow[r, shift right, "g^{-1}"'] & PA \arrow[r, "e^{-1}"] \arrow[l, bend left=60, "g_*"] & PE \arrow[l, bend left=40, "e_*"]
\end{tikzcd}$$ —

— a split coeq

$\Rightarrow$ p. 2 ✓

3) P is faithful.

$\forall f, g : A \to B$ we can construct the composite

$$B \xrightarrow{\eta} PB \underset{g^{-1}}{\overset{f^{-1}}{\rightrightarrows}} PA$$

$\eta : b \mapsto \{b\}$

These transpose to define maps:

$A \times B \longrightarrow \Omega$,

classifying the objects

$(1, f) : A \rightarrowtail A \times B$

$(1, g) : A \rightarrowtail A \times B$

By the universal property of Ω,

if $f^{-1} = g^{-1}$ then $f = g$

and by Ex 16 III

faithful functors reflects monomorphisms and epimorphisms in Set iso $\Longleftrightarrow$ mono + epi

$\Rightarrow$ P preserves iso $\Rightarrow$ p.3 ✓

$\blacksquare$

5.6. Limits and colimits in categories of Algebras

These results apply to:

$\mathbf{Set}_*$, Mon, Grp, Ab, Ring, CRing, R-Mod, Aff_k, $\mathbf{Set}^{BG}$, $\mathbf{Vect}_k^{BG}$, Lattice, $\mathbf{Top}^{CH}$

— monadic over Set (all except Top^{CH} are finitary)

Generally not true for:
Poset, Top, Fields

Lemma: 5.6.1: $U\; \mathcal{A} \to \mathcal{C}$ monadic. Then U reflects isom-s (= is conservative)

□ Sufficient to show that $U^T: \mathcal{C}^T \longrightarrow \mathcal{C}$ has this property. Then the proof is the same as a proof of a fact for groups: any bijective homomorphism is an isom. ∎

Cor 5.6.2: Any bijective continuous map between compact Hausdorff spaces is an isomorphism.

□ $U: \mathrm{Top}^{CH} \to \mathrm{Set}$ sends bij cont function to a

Bijection $\Rightarrow$ isom in Set.

U is monadic $\Rightarrow$

$\Rightarrow$ our original function in Top^{CH} is a homeo

$\blacksquare$

Cor: 5.6.4 Any bij homo in a category of models for an alg theory is an isom

Cor. $\Rightarrow$ Poset and Top are not monadic over Set.

Theorem 5.6.5

A monadic functor $\mathcal{U} : \mathcal{A} \to \mathcal{C}$ creates:

1) all limits $\mathcal{C}$ has

2) all colimits $\mathcal{C}$ has which are preserved by the monad and its square

Cor 5.6.6: The inclusion of a reflective subcategory creates all limits (reflective subcategory = an inclusion functor has a left adjoint

$\Downarrow$

a limit of a diagram in a subcat stays in a subcat)

Cor 5.6.7. Any category which is monadic over Set is complete with limits created by the forgetful functor this is why

$G \times H$ (group) is $G \times H$ (set)

Forgetful functor does not create colimits

Ex 5.6.8 - p-adic numbers

$$\mathbb{Z}_p = \lim_n \mathbb{Z}/p^n\mathbb{Z}$$

(p-prime)

$$\cdots \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z} \twoheadrightarrow \cdots$$

$$\twoheadrightarrow \mathbb{Z}/p^2\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$$

(in the Ring)

U Ring $\longrightarrow$ Sets

creates this

In 3.2.10 – formula

for lim of ω^{op} – indexed

diagrams:

$\{(a_1, a_2, \ldots) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p^2\mathbb{Z} \times$

$\mid a_i \equiv a_{i+1} \pmod{p^i} \}$

U creates limits $\Rightarrow$

$\Rightarrow$ there is a ring

structure on this.

$\mathbb{Z}_p \xrightarrow{proj} \mathbb{Z}/p^n\mathbb{Z}$ — should be a ring hom $\forall n$

$\Rightarrow$ the operations have to be defined componentwise

Coz 5.6.9: Set is cocomplete

$\square$

P: $Set^{op} \to Set$ is monadic $\Rightarrow$ Set^{op} is complete $\Rightarrow$ Set is cocomplete

$\blacksquare$

Cor 5.6.10 $U: R\text{-Mod} \to Ab$

creates all l.n.t.s. Ab admits + Ab is monadic over Set $\Rightarrow$

$\Rightarrow$ Ab is complete $\Leftarrow$ $\Rightarrow$ R-Mod is complete

Ab (and any cat. of models for an alg. theory) is cocomplete

colimits that are not preserved by T are harder

$G * H$ = "words in $UG \sqcup UH$" / (relations in G and relations in H)

not 'very' rigourous

More formal:

$$\mathrm{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{Grp} \quad \perp$$

First approx

$G * H$ is $F(UG \sqcup UH)$

= words in $UG \sqcup UH$

Problem This forgets

relations in G and H,

ie $hgg'h' \mapsto hg''h'$,

$g'' = gg'$

need to impose rel. of G and H -

- These generated by words in G and words in H $\Rightarrow$ elements of

$\mathcal{F}\mathcal{U}G \sqcup \mathcal{U}\mathcal{F}\mathcal{U}H$

want a group $\Rightarrow$

$F(UFUG \cup UFUH) -$

- group of words of words, each subword is entirely in G or H.

$(g_1 g_2 g_3)(h_1 h_2)(g_4)(h_3 h_4 h_5) - \dots (h_i h_j)$

To get $G * H$ impose relations:

G and H are groups

$\rightsquigarrow$ encoded by the multiplication in monad

$\Rightarrow$ $\varepsilon_G : FUG \to G$ =

= take a word and evaluate it

$\varepsilon_H : FUH \to H$

$$
\begin{tikzcd}
\underset{x}{F(UFUG \sqcup UFUH)} \arrow[rr, "F(U\varepsilon_G \sqcup U\varepsilon_H)"] \arrow[rr, bend left=40] \arrow[dr, "F\varkappa"'] \arrow[rr, bend right=70, "g"'] & & \underset{y}{F(UG \sqcup UH)} \\
& FUF(UG \sqcup UH) \arrow[ur, "\varepsilon_{F(UG \sqcup UH)}"'] &
\end{tikzcd}
$$

$\varkappa : (UF-) \sqcup (UF-) \Rightarrow$

$\Rightarrow UF(- \sqcup -)$

define $G * H$ to be the coeq. of

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y \longrightarrow G * H$$

$G * H$ is a quot. of Y by the relations that coeq this

$$G * H = Y / \langle\langle \{ f(x) g(x)^{-1} \mid x \in X \} \rangle\rangle$$

This def. generalizes Prop. 5.6 n -

$\mathcal{C}$ complete, $U: \mathcal{X} \to \mathcal{C}$ - monadic

Then $\mathcal{A}$ is cocomplete

$\Leftrightarrow$ $\mathcal{A}$ has coeq-s,

$\square$

$\Rightarrow$ - obv.

$\Leftarrow$ Replace $\mathcal{A}$ with $\mathcal{C}^T$

and U with U^T.

As above coeqs gives

coproducts.

$$F(UFUX + UFUY) \overset{f}{\underset{g}{\rightrightarrows}} F(UX+UY) \twoheadrightarrow X+Y$$

Coeqs. + coproducts $\Rightarrow$

$\Rightarrow$ all small colimits

$\Rightarrow$ cocomplete

$\blacksquare$

Th 5.6.12:

1) $T \quad \mathcal{C} \to \mathcal{C}$ — finitary monad
2) $\mathcal{C}$ — complete and cocomplete
3) $\mathcal{C}$ — locally small

Then $\mathcal{C}^T$ is complete and cocomplete

__Cor__ 5.6.14 Any category of models for an alg. theory is complete and cocomplete.