Monadic Functors

Recall: (T, η, μ) monad

$$\begin{tikzcd} \mathbb{D} \arrow[rr, "K"] \arrow[dr, shift left] & & \mathcal{C}^T \arrow[dl, shift left] \\ & \mathcal{C} \arrow[ul, shift left] \arrow[ur, shift left] & \end{tikzcd}$$

canonical comparison functor

eg. $\mathbb{Z}[-] : \mathrm{Set} \to \mathrm{Set}$ free abelian gp functor

$$\begin{tikzcd} \mathrm{Set} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Ab} \arrow[l, shift left, "U"] \end{tikzcd}$$

$\mathrm{Set}^{\mathbb{Z}[-]} \cong \mathrm{Ab}$

obj. $(A, \alpha : \mathbb{Z}[A] \to A)$

↳ evaluation map

Def. $\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, shift left] \end{tikzcd}$ adjunction → called monadic if $K : \mathbb{D} \to \mathcal{C}^T$ is an equivalence of categories & strictly monadic if its an isomorphism.

strictly monadic if K admits a left adjoint st. the adjunction is monadic

eg. Kleisli adj.

$\mathrm{Set}^{\partial} \rightleftarrows \mathrm{Set}$

Canonical presentation via free alg.

Motivation: $\mathbb{Z}[-] : \mathrm{Set} \to \mathrm{Set}$

$$\begin{tikzcd} \mathrm{Set} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Ab} \arrow[l, shift left] \end{tikzcd}$$

$A \in \mathrm{Ab}$, $G \subseteq A$

$\mathbb{Z}[G] \longrightarrow A$

if $A = \langle G \rangle$ this is a surjection

R = relations holding in $\mathbb{Z}[G]$ that we map to 0

$$\mathbb{Z}[R] \underset{ev}{\overset{0}{\rightrightarrows}} \mathbb{Z}[G] \twoheadrightarrow A$$

$$A \cong \mathbb{Z}[G] / \langle R \rangle$$

$\hookrightarrow$ not well-behaved under morphisms $\longrightarrow$ not canonical

$G = A$

R = all words

$$\mathbb{Z}[\mathbb{Z}[A]] \underset{\mu_A}{\overset{\mathbb{Z}[\alpha]}{\rightrightarrows}} \mathbb{Z}[A] \overset{\alpha}{\twoheadrightarrow} A$$

This is the canonical presentation

Prop 5.4.3: Let (T, η, μ) be a monad on $\mathcal{C}$ &

$(A, \alpha) \in \mathcal{C}^T$

Then

$$(T^2 A, \mu_{TA}) \underset{\mu_A}{\overset{T\alpha}{\rightrightarrows}} (TA, \mu_A) \overset{\alpha}{\twoheadrightarrow} (A, \alpha)$$

defines a coequaliser.

Def. A split coequaliser is

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[l, bend left=60, "t"] \arrow[r, "h"] & Z \arrow[l, bend left=60, "s"] \end{tikzcd}$$

st. $hs = 1_Z$, $hf = hg$, $gt = 1_Y$, $ft = sh$

Lem. A split coequaliser is a coequaliser & is absolute (preserved by all functors).

Pf. $k : Y \to W$ st $kf = kg$ should factor through h is a split epi. so get

$$\begin{tikzcd} X \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[r, "h"] \arrow[dr, "k"'] & Z \arrow[d, "\gamma"] \\ & & W \end{tikzcd}$$

$\gamma h = k \Rightarrow \gamma = ks$

$ksh = k$ b/c $ksh = kft = kgt = k$

Absolute ?

Def. $U : \mathcal{D} \to \mathcal{C}$

$f, g : x \to y$ in $\mathcal{D}$

1. a U-split coequaliser is $\exists$ extension, $Uf, Ug : Ux \to Uy$, to the split coequaliser

$$\begin{tikzcd} Ux \arrow[r, shift left, "Uf"] \arrow[r, shift right, "Ug"'] & Uy \arrow[r] \arrow[l, bend left, "t"] & h \arrow[l, bend left, "s"] \end{tikzcd}$$

2. U creates coequalisers of U-split pairs if any U-split coequaliser admits a coequaliser in $\mathcal{D}$ st. its image in $\mathcal{C}$ under U is this coequaliser

& any such fork in $\mathcal{D}$ is a coequaliser.

3. U strictly create coeq. of U-split pairs if it creates them & the lift is unique.

Prop. For any monad (T, η, μ) the forgetful functor
$U^T : \mathcal{C}^T \to \mathcal{C}$
strictly creates coeq. of U^T-split pairs.

Pf. $f, g : (A, \alpha) \rightrightarrows (B, \beta)$ st.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] \arrow[l, bend left=60, "t"] & C \arrow[l, bend left=60, "s"] \end{tikzcd}$$

Show
1. (C, γ) alg. structure on C
2. h hom
3. h coeq. in $\mathcal{C}^T$
4. unique

If we apply T we get

$$\begin{tikzcd} TA \arrow[r, shift left, "Tf"] \arrow[r, shift right, "Tg"'] \arrow[d, "\alpha"'] & TB \arrow[r, "Th"] \arrow[d, "\beta"] & TC \arrow[d, dashed] \\ A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"'] & C \end{tikzcd}$$

$\exists ! \gamma \to$ by universal property of coeq. & fact that $h\beta$ coequalises Tf & Tg ie

$h\beta Tf = h\beta Tg$

$h\beta Tf \underset{\text{b/c } f \text{ morp of algs.}}{=} hf\alpha = hg\alpha = h\beta Tg$

(C, γ) alg.?

$$\begin{tikzcd} C \arrow[r, "\eta_C"] \arrow[dr, "1_C"'] & TC \arrow[d, "\gamma"] \\ & C \end{tikzcd} \quad \& \quad \begin{tikzcd} T^2C \arrow[r, "\mu_C"] \arrow[d, "T\gamma"'] & TC \arrow[d, "\gamma"] \\ TC \arrow[r, "\gamma"'] & C \end{tikzcd}$$

commute?

$$\begin{tikzcd} B \arrow[r, "\eta_B"] \arrow[dr, "1_B"'] \arrow[rrr, bend left, "h"] & TB \arrow[d, "\beta"] \arrow[rrrr, bend left, "Th"] & & & C \arrow[r] \arrow[dr, "1_C"'] \arrow[r, phantom, shift right=4, "*"] & TC \arrow[d, "\gamma"] \\ & B \arrow[rrrr, bend right, "h"'] & & & & C \end{tikzcd}$$

prism & all faces commute except possibly $*$

$\gamma \eta_C = 1_C \iff \gamma \eta_C h = 1_C h$ b/c h split epi.

$\gamma \eta_C h = \gamma T h \eta_B = h \beta \eta_B = h 1_B = 1_C h$

So $*$ commutes also.

Proof for square commuting is similar.

Just have to prove h is coeq. in $\mathcal{C}^T$

$k : (B, \beta) \to (D, \delta)$ st $kf = kg$.

Should factorise through C.

$$\begin{tikzcd} A \arrow[r, shift left] \arrow[r, shift right] & B \arrow[r, "h"] \arrow[dr, "k"'] & C \arrow[d, "j"] \\ & & D \end{tikzcd}$$

$j \rightsquigarrow$ exists b/c coeq. in $\mathcal{C}$

need to show j is hom ie $\gamma j = \gamma T j$

Th split epi $\Rightarrow$ $j\gamma Th = jh\beta = k\beta = \delta Tk = \delta Tj Th$

Pf. of canonical presentation (?)

$$\underbrace{(T^2A, \mu_{TA}) \underset{\mu_A}{\overset{T\alpha}{\rightrightarrows}} (TA, \mu_A)}_{} \xrightarrow{\alpha} (A, \alpha)$$

split b/c of

$$\begin{tikzcd} T^2A \arrow[r, shift left, "T\alpha"] \arrow[r, shift right, "\mu_A"'] & TA \arrow[l, bend left=80, "\eta_{TA}"] \arrow[r, "\alpha"] & A \arrow[l, bend left=60, "\eta_A"] \end{tikzcd}$$

$\square$

Corollary: If $\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathbb{D}$ ($\perp$) a monadic adjunction then

1. $U: \mathbb{D} \to \mathcal{C}$ creates coeq split pairs
2. coeq.

$$FUFUD \underset{FU\varepsilon_D}{\overset{\varepsilon_{FUD}}{\rightrightarrows}} FUD \xrightarrow{\varepsilon_D} D$$

Problem: how to recognise monadic functors

Soln:

Thm: $U: \mathbb{D} \to \mathcal{C}$ a right adj. functor is (strictly) monadic iff it (strictly) creates coeq. of U-split pairs.

Pf. ($\Rightarrow$) corollary above

($\Leftarrow$)

$$\begin{tikzcd} \mathbb{D} \arrow[rr, "K"] \arrow[dr, shift left, "U"] & & \mathcal{C}^T \arrow[ll, bend right, dashed, "L"'] \arrow[dl, shift left, "U^T"] \\ & \mathcal{C} \arrow[ul, shift left, "F"] \arrow[ur, shift left, "F^T"] & \end{tikzcd}$$

$LK \cong 1_{\mathbb{D}}$ & $KL \cong 1_{\mathcal{C}^T}$

$U^T K = U$ & $KF = F^T$

Need

$U^T \cong UL$

$F \cong LF^T$

In fact can take $LF^T = F$ ie L acts on free alg. as F does on $\mathcal{X}$

$L(TX, \mu_X) = FX$

$Tf : (TX, \mu_X) \to (TY, \mu_Y)$ map of free alg.

$Lf := Ff$

Need to extend to <u>all</u> alg.

For any alg. (A, α)

$$FUFA \underset{\varepsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA \to L(A, \alpha)$$

exists b/c comes from U-split pair, applying U gives $T^2A \underset{\mu_A}{\overset{T\alpha}{\rightrightarrows}} TA$

$$\begin{tikzcd}
FUFA \arrow[r, shift left, "F\alpha"] \arrow[r, shift right, "\varepsilon_{FA}"'] \arrow[d, "FUFf"'] & FA \arrow[r] \arrow[d, "Ff"] & L(A,\alpha) \arrow[d, dashed, "\exists! Lf"] \\
FUFB \arrow[r, shift left, "F\beta"] \arrow[r, shift right, "\varepsilon_{FB}"'] & FB \arrow[r] & L(B,\beta)
\end{tikzcd}$$

This def's L fully.

$KL \cong 1_C$ & $LK \cong 1_D$

K carries

$$FUFA \underset{\varepsilon_{FA}}{\overset{F\alpha}{\rightrightarrows}} FA$$

to

$$(T^2A, \mu_A) \underset{\mu_A}{\overset{T\alpha}{\rightrightarrows}} (TA, \mu_A)$$

which is U^T-split & as U^T strictly creates coeq. of U^T split pairs

$$Kh(A,\alpha) \cong (A,\alpha)$$

by uniqueness of coeq.

$\Rightarrow KL \cong 1_{\mathcal{C}^T}$

$LK \cong 1_{\mathcal{D}} \quad \forall \; D \in \mathcal{D}$

$$\begin{tikzcd} FUFUD \arrow[r, shift left, "FU\varepsilon_D"] \arrow[r, shift right, "\varepsilon_{FUD}"'] & FUD \arrow[r] \arrow[dr] & LKD \arrow[r] & \text{coeq} \\ & & D \arrow[r] & \text{from cor. also coeq.} \end{tikzcd}$$

$\therefore \; LKD \cong D \;\Rightarrow\; LK \cong 1_{\mathcal{D}}$

eg. $U : \mathcal{C} \to Set$ "monadic over set"

1. $\mathcal{C}$ = Mon, Grp, Ab, Ring, ...
2. $Vect_K$, R-Mod, Aff
3. Lattices, semilattices

Pf. for Mon.

U : Mon $\to$ Set forgetful

$$f, g : M \rightrightarrows M' \quad \text{st} \quad \begin{tikzcd} M \arrow[r, shift left, "f"] \arrow[r, shift right] & M' \arrow[r, "h"] \arrow[l, bend left, "t"] & M'' \arrow[l, bend left, "s"] \end{tikzcd}$$

WTS. have monoid structure on M''

· h hom

• This fork of coeq.

$(-)^2 : \mathrm{Set} \to \mathrm{Set}$

$X \mapsto X \times X$

$$\begin{tikzcd}
M \times M \arrow[r, shift left, "f \times f"] \arrow[r, shift right, "g \times g"'] \arrow[d, "\mu"'] & M' \times M' \arrow[r, "h \times h"] \arrow[d, "\mu'"] & M'' \times M'' \arrow[d, dashed] \\
M \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & M' \arrow[r, "h"'] & M''
\end{tikzcd}$$

$\exists!$ by universal property of coequalisers

↳ this gives mult. on M''

$\square$