INTERMEDIATE CAT THEORY

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1. Recap Lecture

1.1. **Chapter 1.** Skipping basic definitions see Chapter 1 of Context.

1.2. **Chapter 2.** Starting with the Yoneda Lemma.

Paraphrase

"If we want to understand a particular object in a category by understanding its relationship to all other objects"

Understanding representable functors first

Let $\mathbb{C}$ be a locally small category (recall: that the class of all morphisms between pairs of objects are sets)

Let $\mathsf{F} : \mathbb{C} \longrightarrow \mathbb{S}\mathsf{et}$ be a (covariant) functor if there is exists a $\mathsf{c} \in \mathbb{C}$ such that

$$\mathbb{C}(\mathsf{c}, -) \cong \mathsf{F}$$

we say F is a representable functor.

Let $\mathsf{F} : \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbb{S}\mathsf{et}$ be a (contravariant) functor if there exists a $\mathsf{c} \in \mathbb{C}$

$$\mathbb{C}(-, \mathsf{c}) \cong \mathsf{F}$$

we also say F is a representable functor.

Examples:

- Let $\mathbf{1} : \mathbb{S}\mathsf{et} \longrightarrow \mathbb{S}\mathsf{et}$ be the identity functor, is representable by the singleton set $\{*\}$, $\mathbb{S}\mathsf{et}(\{*\}, -)$.
- Let $\mathcal{P} : \mathbb{S}\mathsf{et}^{\mathrm{op}} \longrightarrow \mathbb{S}\mathsf{et}$ be the contravariant power set functor. This functor is representable by the 2 element set $\{0, 1\}$, $\mathbb{S}\mathsf{et}(-, \{0, 1\})$.
- Let $\mathsf{U} : \mathbb{G}\mathsf{rp} \longrightarrow \mathbb{S}\mathsf{et}$ be the forgetful functor, this is represented by $\mathbb{Z}$ i.e. $\mathbb{G}\mathsf{rp}(\mathbb{Z}, -)$.
- $\Delta_\emptyset : \mathbb{S}\mathsf{et}^{\mathrm{op}} \longrightarrow \mathbb{S}\mathsf{et}$ the constant functor that sends all sets to the empty set is not representable. Exercise: Hint: Try to represent it and use that $\emptyset$ is the initial object in $\mathbb{S}\mathsf{et}$.

The Yoneda Lemma proper:

Let $\mathbb{C}$ be a locally small category, then for a functor $\mathsf{F} : \mathbb{C} \longrightarrow \mathbb{S}\mathsf{et}$ we have

$$[\mathbb{C}, \mathbb{S}\mathsf{et}](\mathbb{C}(\mathsf{c}, -), \mathsf{F}) \cong \mathsf{Fc}$$

natural in F and c. The contravariant version:

$$[\mathbb{C}^{\mathrm{op}}, \mathbb{S}\mathsf{et}](\mathbb{C}(-, \mathsf{c}), \mathsf{F}) \cong \mathsf{Fc}$$

again natural in F and c.

Lets apply this immediately, to $\mathcal{P} : \mathbb{S}\mathsf{et}^{\mathrm{op}} \longrightarrow \mathbb{S}\mathsf{et}$, we know that $\mathcal{P} \cong \mathbb{S}\mathsf{et}(, \{0, 1\})$.

$$[\mathbb{S}\mathsf{et}^{\mathrm{op}}, \mathbb{S}\mathsf{et}](\mathbb{S}\mathsf{et}(, \{0, 1\}), \mathcal{P}) \cong \mathcal{P}(\{0, 1\})$$

Recall that $\mathcal{P}(\{0, 1\})$ is a 4 element set $\{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.

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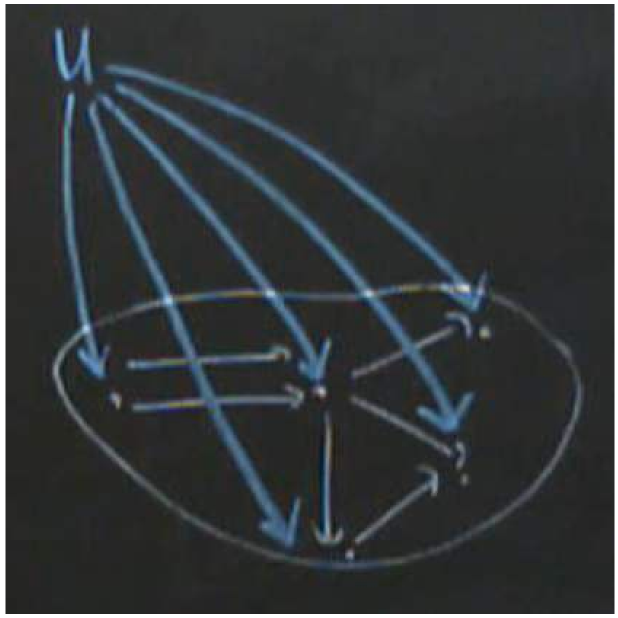

Figure 1. A Cone

Jump ahead to the Yoneda Embedding:
$\mathbb{C}(c, -) : \mathbb{C} \longrightarrow \mathbb{S}\mathrm{et}$, consider (you can use the katakana 'yo' symbol)

$$\mathbb{C}^{\mathrm{op}} \longrightarrow [\mathbb{C}, \mathbb{S}\mathrm{et}]$$

$$c \mapsto \mathbb{C}(c, -)$$

a contravariant embedding meaning that it is full and faithful. The functor $\mathbb{C}(-, c) : \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbb{S}\mathrm{et}$. The Covariant embedding is then

$$\mathbb{C} \longrightarrow [\mathbb{C}^{\mathrm{op}}, \mathbb{S}\mathrm{et}]$$

$$c \mapsto \mathbb{C}(-, c)$$

1.3. **Chapter 3.** Limits and Colimits

$$\begin{tikzcd}
 & W \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\
X \arrow[r] & X \times Y & Y \arrow[l]
\end{tikzcd}$$

Again let $\mathbb{C}$ be locally small, and let $F : \mathcal{I} \longrightarrow \mathbb{C}$ we need

In the figure we use the u to be the zenith of our cone and the white arrows to be the diagram, the image of F in $\mathbb{C}$.

Jumping back to beginning of limits and colimits. We view limits as representing objects of the functor Cone$(-, F)$ cones with diagram F, and Cocone$(F, -)$. See figure 1.

More about products, the limits out of the discrete two object category is a product. Coproducts are colimits out of the same category.

Pullbacks are limits out of the "universal cospan category". Pushouts are colimits out of the "universal span category".

Exercises:

- What is the coproduct of two sets? Think sum

$$
\begin{tikzcd}[column sep=small]
\mathsf{C} \arrow[dddd, "F"'] & \ni & p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & c'' \arrow[d] & & p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & c'' \arrow[d] & & p \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & c'' \arrow[d] \\
& & c' \arrow[r] & c & & c' \arrow[r] & c & & c' \arrow[r] & c \\
& & \text{preserves} \rotatebox{-90}{$\rightsquigarrow$} & & & \text{reflects} \rotatebox{90}{$\rightsquigarrow$} & & & \text{creates} \rotatebox{90}{$\rightsquigarrow$} & \\
& & Fp \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Fc'' \arrow[d] & & Fp \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Fc'' \arrow[d] & & d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Fc'' \arrow[d] \\
\mathsf{D} & \ni & Fc' \arrow[r] & Fc & & Fc' \arrow[r] & Fc & & Fc' \arrow[r] & Fc
\end{tikzcd}
$$

FIGURE 2. Cospan in $\mathbb{C}$: $c' \longrightarrow c \longleftarrow c''$

- What is the product of two groups? Don't think
- What is the coproduct of two groups? Think is the coproduct of underlying sets 'big enough' to be a group?
- What is a pullback in $\mathbb{S}\mathsf{et}$? Think kernel pair
- What is the limit of a diagram of this shape in $\mathbb{G}\mathsf{rp}$:

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

Group structure on this is easy to write down? Bonus: Write this down.

Question (Steven): Is it not strange to just write down $X + Y$ a sum of two sets if any other set would also be a coproduct? Answer: Actually we have extra data that is suppressed, the coproduct is the unique cocone but we are usually considering just the nadir of the cocone.

1.4. Some quick ideas on special functors.

$$\mathbb{J} \xrightarrow{F} \mathbb{C} \xrightarrow{G} \mathbb{D}$$

Consider also $H : \mathbb{J} \longrightarrow \mathbb{D}$ preserves limits: ... see figure 2
Finite limtis commute with filtered colimts.

1.5. Adjunctions. $R : \mathbb{D} \leftrightarrow \mathbb{C} : L$

Functors back and forth.

Aside on categories of categories as a 2-category, $\mathbb{CAT}$. An isomorphism is not very interesting, but an equivalence is! (Recall full, faithful, almost surjective on objects)

An equivalence would be the data

$$\eta : \mathbf{1} \longrightarrow RL$$

$$\epsilon : LR \longrightarrow \mathbf{1}$$

where each natural transformation is a natural isomorphism. This is a little too strong though.

We instead ask for the following triangle

$$
\begin{tikzcd}
R \arrow[r, Rightarrow, "\eta_R"] \arrow[dr, Rightarrow, "id_R"'] & RLR \arrow[d, Rightarrow, "R\epsilon"] \\
& R
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[d, Rightarrow, "L\eta"'] \arrow[dr, Rightarrow, "id_L"] & \\
LRL \arrow[r, Rightarrow, "\epsilon L"'] & L
\end{tikzcd}
$$

If this situation happens then we write: $\mathsf{R} \vdash \mathsf{L}$ and call R right adjoint to L and L is left adjoint to R.

We can also write this as an isomorphism of hom-sets:

$$\mathbb{C}(c, Rd) \cong \mathbb{D}(Lc, d).$$

How do we get the η and ϵ from this?

Exercise: Write down what is needed for η and ϵ at an object c. Hint: use $d = Lc$ and consider the proof of the Yoneda Lemma.

Example:

$U : \mathbb{G}\mathsf{rp} \leftrightarrow \mathbb{S}\mathsf{et} : F$, where U is the forgetful functor and F is the free group making functor.