§ t-structures

Prp Let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a t-structure on $\mathcal{T}$.

i) The inclusion $\mathcal{T}^{\leq n} \to \mathcal{T}$ admits a right adjoint
$${}^t\tau^{\leq n} : \mathcal{T} \to \mathcal{T}^{\leq n}$$

ii) The inclusion $\mathcal{T}^{\geq n} \to \mathcal{T}$ admits left adjoint
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These maps are called truncation.

Lemma $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ t-structure $\forall a \leq b$ we have nat. trans.
$${}^t\tau^{\leq a} \circ {}^t\tau^{\leq b} \xrightarrow{\sim} {}^t\tau^{\leq a}$$
$${}^t\tau^{\geq b} \xrightarrow{\sim} {}^t\tau^{\geq b} \circ {}^t\tau^{\geq a}$$
$${}^t\tau^{\geq a} \circ {}^t\tau^{\leq b} \xrightarrow{\sim} {}^t\tau^{\leq b} \circ {}^t\tau^{\geq a}.$$

Def. $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ t-structure $\mathcal{C} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ the heart
the 0^{th} cohomology wrt the t-structure is
$${}^tH^0 : {}^t\tau^{\leq 0} \circ {}^t\tau^{\geq 0} : \mathcal{T} \to \mathcal{C}$$

$\forall n \in \mathbb{Z}$ define ${}^tH^n : \mathcal{T} \to \mathcal{C}$ $\quad {}^tH^n(X) = {}^tH^0(X[n])$

This is a cohomological functor.
i.e. $X \to Y \to Z \to X[1]$ dist. tri. $\Rightarrow$ $H^n(X) \to H^n(Y) \to H^n(Z)$ exact.

Def $\mathcal{T}_1, \mathcal{T}_2$ triangulated categories w/ t-structures $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$. A triangulated functor $F : \mathcal{T}_1 \to \mathcal{T}_2$ left t-exact if $F(\mathcal{T}_1^{\geq 0}) \subset \mathcal{T}_2^{\geq 0}$. right t-exact if $F(\mathcal{T}_1^{\leq 0}) \subset \mathcal{T}_2^{\leq 0}$. t-Exact if both.

Lemma With $\mathcal{T}_1, \mathcal{T}_2, F$ as above

i) F left exact $\Rightarrow$ ${}^tH^0 \circ F : \mathcal{C}_1 \to \mathcal{C}_2$ left exact

ii) F right exact $\Rightarrow$ ${}^tH^0 \circ F : \mathcal{C}_1 \to \mathcal{C}_2$ right exact.

Rmk $F: \mathcal{T}_1 \to \mathcal{T}_2$ as above. We say F has cohom. dim. $\leq d$ if $F(\mathcal{T}_1^{\leq 0}) \subseteq \mathcal{T}_2^{\leq d}$. In this situ.

$${}^tH^{n+d}(F(X)) \cong {}^tH^d(F({}^tH^n(X))). \ \forall X \in \mathcal{T}_1^{\leq n}.$$

Thm. $\mathcal{T}$ triangulated with filtration. $\mathcal{C}$ heart of bounded t-structure. $\exists$ t-exact tri. functor

$$\mathrm{real}: D^b(\mathcal{C}) \to \mathcal{T}$$

whose restr. to $\mathcal{C} \subset D^b(\mathcal{C})$ is $\mathcal{C} \to \mathcal{T}$ inclusion.

We get natural iso.

$$H^i(X) \cong {}^tH^i(\mathrm{real}(X)) \quad \forall X \in D^b(\mathcal{C})$$

§ Perverse t-structure

Let X/k be a variety.

Def The perverse t-structure on ~~X~~ on X is a t-structure on $D^b_c(X,k)$ given by

$$^pD^b_c(X,K)^{\leq 0} = \{\mathcal{F} \mid \forall i \ \dim\operatorname{supp}(\mathcal{H}^i(\mathcal{F})) \leq -i\}$$
$$^pD^b_c(X,k)^{\geq 0} = \{\mathcal{F} \mid \forall i \ \dim\operatorname{supp}(\mathcal{H}^i(\mathbb{D}\mathcal{F})) \leq -i\}$$

where $\mathbb{D}\mathcal{F}$ denotes the verdier dual.

$$\operatorname{Perv}(X,k) = {}^pD^b_c(X,k)^{\leq 0} \cap {}^pD^b_c(X,k)^{\geq 0}$$

Def Let $\{X_i\}_{i\in I}$ be a good stratification of X

$$^pD^b_I(X,k)^{\geq 0} = {}^pD^b_c(X,k)^{\geq 0} \cap D^b_I(X,k)$$
$$^pD^b_I(X,k)^{\leq 0} = {}^pD^b_c(X,k)^{\leq 0} \cap D^b_I(X,k)$$

Thm Both of the above are bounded t-structures.

From the theory of t-structures we get

$$^pH^n : D^b_c(X,k) \to \operatorname{Perv}(X,k)$$
$$^pH^n(F) = {}^p\tau^{\leq 0}\,{}^p\tau^{\geq 0}(F[n]).$$

Def The n^{th} perverse cohomology of $F \in D^b_c(X,k)$ is $^pH^n(F)$.

A distinguished triangle

$$F^\bullet \to G^\bullet \to H^\bullet \to$$

gives us long exact seq.

$$^pH^{n-1}(H^\circ) \to {}^pH^n(F^\circ) \to {}^pH^n(G^\circ) \to {}^pH^n(H^\circ) \to \cdots$$

Rmk. $U \in \{\text{open subsets of } X\} \mapsto \operatorname{Perv}(U,k)$ defines a stack.

i.e. $\{U_i\}$ open cover of U if we have $F_i \in \operatorname{Perv}(U_i,k)$ with $F_{i,j} \cong F_{j,i}$ $\exists!\, F \in \operatorname{Perv}(U,k)$ so $F|_{U_i} = F_i$

$\rightarrow$ why we call it a sheaf.

Prop. X smooth alg. variety, L local system on X^{an}, $L[d_X] \in Perv(X, k)$.

Prop. Verdier duality $\mathbb{D} : D^b_c(X) \to D^b_c(X)^{op}$ is t-exact and induces exact
$$\mathbb{D}_X : Perv(X, k) \to Perv(X, k)^{op}.$$

Prop. $\iota : Z \hookrightarrow X$ embedding of closed subvariety we get an equivalence of categories
$$Perv(Z, k) \xrightarrow{\sim} \{F \in Perv(X, k) \mid supp\, F \subset Z\}.$$

Def Given X, Y varieties and $F : D^b_c(X) \to D^b_c(Y)$ a functor of triangulated categories. we get ${}^pF : Perv(X, k) \to Perv(Y, k)$
$$Perv(X, k) \hookrightarrow D^b_c(X) \xrightarrow{F} D^b_c(Y) \xrightarrow{{}^pH^0} Perv(Y, k)$$

Facts If $f : X \to Y$ proper morphism of varieties
$${}^pf^{-1}, {}^pf^! : Perv(Y, k) \to Perv(X, k)$$
$${}^pRf_*\ {}^pRf_! : Perv(X, k) \to Perv(Y, k)$$

Def $F^\bullet \in Perv(X, k)$ $\quad$ $Supp\, F^\bullet$ = complement of largest open subset $U \hookrightarrow X$ s.t. $F^\bullet|_U = 0$

Lemma If $0 \to F^\bullet \to G^\bullet \to H^\bullet \to 0$ exact seq. of perverse sheaves on X
$$Supp\, G^\bullet = supp\, F^\bullet \cup supp\, H^\bullet.$$

Decomposition Theorem

Thrm Let $f: X \to Y$ be proper map of varities w/ X non-singular. There exists finitely many (Y_a, L_a, d_a) Y_a - locally closed smooth irred. alg. ~~structures~~ subvarities, L_a - local system on Y_a, d_a - integer, such that $\forall$ euclidean opens $U \subseteq Y$

$$H^r(f^{-1}U) \simeq \bigoplus_a IH^{r-d_a}(U \cap \overline{Y}_a, L_a)$$