Correction:

P is projective $\Leftrightarrow$ for all diagrams

$$\begin{tikzcd} & P \arrow[dl, dashed, "\psi"'] \arrow[d, "\varphi"] \arrow[dr, "0"] & \\ M' \arrow[r, "i"] & M \arrow[r, "j"] & M'' \end{tikzcd}$$

where the solid arrows commute and $\ker(j) = \operatorname{im}(i)$ there exists a dotted arrow making the whole diagram commute.

Recall exactness & short exact sequences !

Lemma: Free modules are projective.

Proof 1: Let X be a set, and $F(X)$ be the free R-module over it, which means $\operatorname{Hom}_{R\mathrm{Mod}}(F(X), M) \cong \operatorname{Hom}_{\mathrm{Set}}(X, U(M))$ naturally in X and M

$\varepsilon : FU \Rightarrow 1_{R\mathrm{Mod}}$

$$\varphi \longmapsto U(\varphi) \circ \eta_X \quad \text{for } \eta : 1_{\mathrm{Set}} \Rightarrow UF$$

$$\varepsilon_M \circ F(\psi) \longleftarrow \psi$$

$\varepsilon_M : FU(M) \to M$ induced by the identity on the generating set M

$\eta_X : X \to UF(X)$ is the inclusion

where $U(M)$ is the underlying set of M.

To show that $F(X)$ is projective we must show that we can solve the following "lifting problem":

$$\begin{tikzcd} & F(X) \arrow[dl, dashed, "\psi"'] \arrow[d, "\varphi"] \arrow[dr, "0"] & \\ M' \arrow[r, "i"] & M \arrow[r, "j"] & M'' \end{tikzcd}$$

$j \circ \varphi = 0 \Rightarrow \operatorname{im}(\varphi) \subseteq \ker(j) \overset{\text{exactness}}{=} \operatorname{im}(i)$

$\Rightarrow \forall x \in X \ \exists y \in M'$ s.t. $U(i)(x) = (U(\varphi) \circ \eta_X)(y)$

set $\psi' : X \to U(M')$ and let $\psi = \varepsilon_M \circ F(\psi')$
$x \mapsto y$

denote $\varphi' = U(\varphi) \circ \eta_X$, then $\varepsilon_M \circ F(\varphi') = \varphi$ and

$$\begin{tikzcd} & X \arrow[dl, "\psi'"'] \arrow[d, "\varphi'"] \\ U(M') \arrow[r, "U(i)"] & U(M) \end{tikzcd}$$

commutes by definition.

Now we have

$$\begin{tikzcd}
& F(X) \arrow[dl, "F(\varphi')"'] \arrow[d, "F(\varphi)"] \arrow[dl, phantom, shift left=4, "\circlearrowright"] & \\
FU(M') \arrow[r, "FU(i)"] \arrow[d, "\varepsilon_{M'}"'] \arrow[dr, phantom, "\circlearrowright"] & FU(M) \arrow[d, "\varepsilon_M"] \\
M' \arrow[r, "i"] & M
\end{tikzcd}$$

commutes by functoriality of F

commutes by definition of natural transformations.

$\square$

<u>Note</u>: we made a choice of $m' \in i^{-1}(\{\varphi(1)\})$ so we have NO uniqueness here!

<u>Proof 2</u>:

A more abstract way to say this: in Set every object is projective in the sense that $\forall X \in$ Set the lifting problem

$$\begin{tikzcd}
& X \arrow[dl, dashed] \arrow[d] \\
A \arrow[r, two heads] & B
\end{tikzcd}$$

can be solved. (axiom of choice)

$F: \mathrm{Set} \to {}_R\mathrm{Mod}$ is a left adjoint to U which preserves epimorphisms. Let $\varepsilon: FU \to 1$ be the counit and $\eta: 1 \to UF$ be the unit.

So consider

$$\begin{tikzcd}
& F(X) \arrow[dl, dashed] \arrow[d, "h"] \\
A' \arrow[r, two heads] & B'
\end{tikzcd}$$

apply U:

$$\begin{tikzcd}
& UF(X) \arrow[dl, dashed] \arrow[d] \\
U(A') \arrow[r, two heads] & U(B')
\end{tikzcd}$$

we get a map

$$\begin{tikzcd}
& X \arrow[d, "\eta_X"] \arrow[ddl, dotted] \\
& UF(X) \arrow[d] \\
U(A') \arrow[r, two heads] & U(B')
\end{tikzcd}$$

if X is projective we have some dotted $f: X \to U(A')$ solving the lifting problem

applying F we get

$$\begin{tikzcd}
 & F(X) \arrow[ddl, "F(f)"'] \arrow[d, "F(\eta_X)"] \\
 & FUF(X) \arrow[d, "FU(h)"] \\
FU(A') \arrow[r, "FU(g)"] \arrow[d, "\varepsilon_{A'}"'] \arrow[dr, phantom, "\circlearrowleft"] & FU(B') \arrow[d, "\varepsilon_{B'}"] \\
A' \arrow[r, two heads, "g"] & B'
\end{tikzcd}$$

the map $\eta_{B'} \circ FU(h) \circ F(\varepsilon_X)$ is the map h gets sent to via

$$\begin{array}{ccccc}
\mathrm{Hom}(F(X), B') & \cong & \mathrm{Hom}(X, U(B')) & \cong & \mathrm{Hom}(F(X), B') \\
\in & & & & \\
h & \longmapsto & U(h) \circ \eta_X & \longmapsto & \varepsilon_{B'} \circ FU(h) \circ F(\eta_X)
\end{array}$$

So our lemma follows from the following to claims:

① A left adjoint to a functor that preserves epimorphisms preserves projectives.

② In the category Set every object is projective. □

Proof 3:

- In ${}_R\mathrm{Mod}$ a free module on the set X is isomorphic to the direct sum $\bigoplus_X R$.

- The direct sum of two projectives is projective:
$$\mathrm{Hom}(P \oplus Q, -) \cong \mathrm{Hom}(P, -) \times \mathrm{Hom}(Q, -)$$
naturally and direct products (even infinite ones!) preserve exact sequences.

- R is projective. □

Proposition: Projective resolutions of modules are unique up to unique equivalence in the homotopy category, more precisely let $M \in {}_R\mathrm{Mod}$, $P_\bullet \to M$ and $Q_\bullet \to M$ be projective resolutions of M. Now there exists some $h : P_\bullet \to Q_\bullet$ augmentation preserving chain homotopy equivalence and any other $h' : P_\bullet \to Q_\bullet$ augmentation preserving chain homotopy equivalence is chain homotopic to h.

Proof:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, dashed, "h_2"] & P_1 \arrow[r, "d_1^P"] \arrow[d, dashed, "h_1"] \arrow[dl, "s_1"'] & P_0 \arrow[r, "d_0^P"] \arrow[d, dashed, "h_0"] \arrow[dl, "s_0"'] & M \arrow[r] \arrow[d, equal] \arrow[dl, "0"'] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "d_0^Q"] & M \arrow[r, "d_{-1}^Q"] & 0
\end{tikzcd}$$

h_0 exists by projectivity (as d_0^Q is surjective)

h_1 exists similarly as

$\underbrace{h_0 \circ d_1^P}_{} \circ d_0^Q = 0$

apply to this the

$\vdots$

We get $h : P_\bullet \to Q_\bullet$

however not uniquely:

assume $h, h' : P_\bullet \to Q_\bullet$ are chain maps, consider $(h - h') : P_\bullet \to Q_\bullet$ now $d(h_0 - h_0') = 0 \underset{\text{exactness}}{\Rightarrow} \exists s_0 : P_0 \to Q_1$ s.t. $d_0^Q \circ s_0 = h_0 - h_0'$ then

$$d(h_1 - h_1') = (h_0 - h_0')d = d s_0 d$$

$$\Rightarrow d((h_1 - h_1') - s_0 d) = 0 \underset{\text{exactness}}{\Rightarrow} \exists s_1 : P_1 \to Q_2 \text{ s.t. } ds_1 + s_0 d = h_1 - h_1'$$

Now we start the induction: $d(h_2 - h_2') = (h_1 - h_1')d = (ds + sd)d =$
$= dsd$

and thus $d((h_2 - h_2') - sd) = 0 \Rightarrow \exists s_2 : P_2 \to Q_3$ s.t.
$ds_2 + s_1 d = h_2 - h_2'$

and via induction we have that any two h_1 and h_2 are chain homotopic.

Now simply exchange the role of P and Q to get some $g : Q_\bullet \to P_\bullet$ and then by the same argument as before $gh \simeq id$ and $hg \simeq id$. $\square$

Quick rundown of useful facts about projective modules:

- let $0 \to A \to B \to C \to 0$ be a SES. If C is projective $\Rightarrow$ the SES splits:

$$\begin{tikzcd} & & & C \arrow[dl, dashed, "s"'] \arrow[d, equal] & \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r, "f"] & C \arrow[r] & 0 \end{tikzcd} \qquad f \circ s = id$$

and such an SES splits $\Leftrightarrow B \cong A \oplus C$ an application of the 5-lemma:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A \oplus C \arrow[r] \arrow[d, "g \oplus s"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "g"] & B \arrow[r, "f"] & C \arrow[r] & 0 \end{tikzcd}$$

- $P \in {}_R Mod$ is projective $\Leftrightarrow \exists Q \in {}_R Mod$ s.t. $P \oplus Q$ is free

$\Rightarrow$ $0 \to ker(f) \to F \xrightarrow{f} P \to 0$ $\quad F \cong ker(f) \oplus P$

$\Leftarrow$
$$\begin{tikzcd} P \oplus Q \arrow[r, "\pi"'] \arrow[d, dashed] & P \arrow[l, bend right, "\iota"'] \arrow[d] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

- A flat module is a module $M \in {}_R Mod$ such that $- \otimes_R M : {}_R Mod \to Ab$ is exact

projective $\Rightarrow$ flat $\quad \exists Q$ s.t. $\quad P \oplus Q$ is free i.e. $P \oplus Q \cong \bigoplus_I R$

$$- \otimes_R (P \oplus Q) \cong (- \otimes_R P) \oplus (- \otimes_R Q)$$
$$\cong$$
$$- \otimes_R \left(\bigoplus_I R\right) \cong \underbrace{\bigoplus_I (- \otimes_R R)}_{\text{exact}}$$

$\Rightarrow - \otimes_R P$ and $- \otimes_R Q$ is exact

- tensor – hom adjunction: Let $M \in {}_R\mathrm{Mod}$

we have the adjunction

$$\begin{tikzcd} {}_R\mathrm{Mod} \arrow[r, bend right, "\mathrm{Hom}_R(M,-)"'] \arrow[r, phantom, "\perp"] & \mathrm{Ab} \arrow[l, bend right, "M \otimes_{\mathbb{Z}} -"'] \end{tikzcd}$$

note that M has the structure of a left R-module and as an abelian group the structure of a right $\mathbb{Z}$-module.

$M \otimes_{\mathbb{Z}} -$ has the structure of an $(R,\mathbb{Z})$-bimodule i.e. a left R-module.

This means in particular

$$\mathrm{Hom}_R(M \otimes_{\mathbb{Z}} A, N) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathrm{Hom}_R(M,N))$$

<u>Remarks</u>: We care more about projective than free modules because it is not that clear how to define free objects in a general abelian category (to have something like that we need some kind of forgetful functor and then some luck for us to get a free-forgetful adjunction) however projectives are defined via a universal property that makes sense in not just abelian categories (instead of epi maps whose cokernel is zero i.e. surjective maps we can use the notion of epimorphisms).

Chapter VI The homology of a group

Let $R \in CAlg(Ab)$ $T: {}_R Mod \to Ab$ additive functor

i.e. $Hom_R(M,N) \to Hom_{\mathbb{Z}}(TM, TN)$ is a map in Ab

Let $P_\bullet \to M$ be a projective resolution and apply T term-wise, $H_n(TP_\bullet)$ measures the failure of exactness in T. (If T is exact clearly

$$H_n(TP_\bullet) \cong \begin{cases} TM & \text{for } n=0 \\ 0 & \text{otherwise} \end{cases}$$

Remark: Let $\mathcal{A}$ be an abelian category, $K(\mathcal{A})$ be the category of chain complexes built of objects in $\mathcal{A}$ up to homotopy with maps up to homotopy. If $\mathcal{A}$ has enough projectives then we have a well-defined functor

$$\mathcal{A} \xrightarrow{P(-)} K(\mathcal{A})$$
$$A \longmapsto \text{projective resolution of } A$$

Let $F: \mathcal{A} \to \mathcal{B}$ be an additive functor, then $L_n F$ the n^{th} derived functor of F is the composition

$$\mathcal{A} \xrightarrow{P(-)} K(\mathcal{A}) \xrightarrow{K(F)} K(\mathcal{B}) \xrightarrow{H_n} \mathcal{B}.$$

Let M be a G-module. $M_G = M / \langle gm - m \mid m \in M, g \in G \rangle$

$$M^G = \{ m \in M \mid \forall g \in G \ gm = m \}$$

Proposition: $M_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} M$ where $\mathbb{Z} \in \mathrm{Mod}_{\mathbb{Z}G}$ via the trivial action.

Proof:

$$\begin{aligned} M_G &\longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}G} M \\ [m] &\longmapsto (1 \otimes m) \end{aligned}$$

well-defined:

$[gm - m] \longmapsto (1 \otimes (gm - m))$

$$\begin{aligned} 1 \otimes (gm - m) &= 1 \otimes gm - 1 \otimes m = g.1 \otimes m - 1 \otimes m \\ &= 1 \otimes m - 1 \otimes m \\ &= 0 \end{aligned}$$

define the inverse as:

$$\begin{aligned} \mathbb{Z} \otimes_{\mathbb{Z}G} M &\longrightarrow M_G \\ a \otimes m &\longmapsto a[m] \end{aligned}$$

$\square$

Corollary: $(-)_G : {}_{\mathbb{Z}G}\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$ is right exact (as the tensor is a left adjoint)

If $F \cong \bigoplus_I \mathbb{Z}G$ then $(F_G) \cong \bigoplus_I (\mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}G) \cong \bigoplus_I \mathbb{Z}$

Proposition: Let X be a free G-complex and let Y be the orbit complex X/G. Then $C_*(Y) \cong C_*(X)_G$.

Proof: $X \overset{q}{\twoheadrightarrow} X/G$ induces some $q_* : C_*(X) \to C_*(Y)$ on $\mathbb{Z}G$-modules, as $q_*(g.e_\alpha) = q_*(e_\alpha)$ we get that q_* factors as

$$\begin{tikzcd} C_*(X) \arrow[rr, "q_*"] \arrow[dr] & & C_*(Y) \\ & C_*(X)_G \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} C_n(X) \cong \mathbb{Z}[n\text{-cells}] \cong \bigoplus_{\text{orbits of } n\text{-cells}} \mathbb{Z}G \arrow[rr] \arrow[dr] & & C_n(Y) \cong \mathbb{Z}[\text{orbits of } n\text{-cells}] \cong \bigoplus_{\text{orbits of } n\text{-cells}} \mathbb{Z} \\ & C_n(X)_G \arrow[ur] & \end{tikzcd}$$

$$\begin{aligned} C_n(X)_G &\cong \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}[n\text{-cells}] \\ &\cong \mathbb{Z} \otimes_{\mathbb{Z}G} \Big(\bigoplus_{\text{orbits of } n\text{-cells}} \mathbb{Z}G\Big) \\ &\cong \bigoplus_{\text{orbits of } n\text{-cells}} \mathbb{Z} \end{aligned}$$

$\square$

Def.: Let G be a group and $F_\bullet \xrightarrow{\varepsilon} \mathbb{Z}$ a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. The n^{th} homology group of G is

$$H_i(G) \cong H_i(F_G) \qquad C_*(EG) \qquad C_*(EG/G) = C_*(BG)$$

$$\overset{\shortparallel}{\mathbb{F}} \qquad \overset{\cup}{F_G}$$

Example: Take C_n the cyclic group of order n. Recall: $\mathbb{Z}C_n \cong \mathbb{Z}[t]/(t^n-1)$

We have the resolution

$$C_n = (\{1, t, \dots, t^{n-1}\}, \times)$$

$$\begin{array}{ccccccccccc} \dots \to \mathbb{Z}C_n & \longrightarrow & \mathbb{Z}C_n & \longrightarrow & \mathbb{Z}C_n & \xrightarrow{\cdot(t-1)} & \mathbb{Z}C_n & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ \wr\shortparallel & & \wr\shortparallel & & \wr\shortparallel & & \wr\shortparallel & & \shortparallel & & \\ \dots \to \mathbb{Z}[t]/(t^n-1) & \longrightarrow & \mathbb{Z}[t]/(t^n-1) & \longrightarrow & \mathbb{Z}[t]/(t^n-1) & \longrightarrow & \mathbb{Z}[t]/(t^n-1) & \to & \mathbb{Z} & \longrightarrow & \end{array}$$

$1 \longmapsto 1$

$1 \longmapsto (t-1) \qquad 1 \longmapsto (t-1)$ module maps!

$1 \longmapsto (1+t+\dots+t^{n-1}) \qquad t \mapsto 1$

$p \mapsto 0 \iff p = (t-1)q$

$\Rightarrow \ker(t \mapsto 1) = (t-1)$

$$p(t)(t-1) = 0 \iff (t^n-1)q(t) = p(t)(t-1)$$

$$\iff p(t) = (1+t+\dots+t^{n-1})\,p'(t)$$

applying $\mathbb{Z} \otimes_{\mathbb{Z}C_n} -$ we get the chain complex

we delete the augmentation!

$$\dots \to \mathbb{Z} \otimes_{\mathbb{Z}C_n} \mathbb{Z}C_n \to \mathbb{Z} \otimes_{\mathbb{Z}C_n} \mathbb{Z}C_n \to \mathbb{Z} \otimes_{\mathbb{Z}C_n} \mathbb{Z}C_n \to 0$$

$1 \otimes 1 \longmapsto 1 \otimes (1+t+\dots+t^{n-1})$

$= (1\otimes 1 + 1 \otimes t + \dots + 1 \otimes t^{n-1})$

$= (1\otimes 1 + t.1\otimes 1 + \dots + t^{n-1}.1\otimes 1)$

$= n \otimes 1$

$1 \otimes 1 \longmapsto 1 \otimes (t-1)$

$= 1\otimes t - 1 \otimes 1$

$= t.1\otimes 1 - 1 \otimes 1$

$= 0$

Thus we get the chain complex $\dots \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$.

For any group G we can take the standard resolution $F_\bullet \to \mathbb{Z}$.
We write $C_*(G)$ for the chain complex F_G.
Explicit description for $C_*(G)$:

<u>homogeneous chain complex of G:</u>
on the $(n+1)$-tuples $(g_0,\dots,g_n)$ we define the equivalence relation
$$(g_0,\dots,g_n) \sim (gg_0,\dots,gg_n) \quad \forall g\in G$$
denote by $[g_0,\dots,g_n]$ the equivalence class of $(g_0,\dots,g_n)$.

$C_n(G)$ is generated as a $\mathbb{Z}$-module by the equivalence classes $[g_0,\dots,g_n]$ and $\partial: C_n(G) \to C_{n-1}(G)$ is given by $\sum_{i=0}^{n}(-1)^i d_i$ where
$$d_i([g_0,\dots,g_n]) = [g_0,\dots,\hat{g}_i,\dots,g_n]$$

<u>non-homogeneous description:</u> $\leadsto$ before this was a $\mathbb{Z}G$-basis & now it is a $\mathbb{Z}$-basis !
C_n has a $\mathbb{Z}$-basis consisting of n-tuples $[g_1|\dots|g_n]$
and $\partial = \sum_{i=0}^{n}(-1)^i d_i$ where
$$d_i[g_1|\dots|g_n] = \begin{cases} [g_2|\dots|g_n] \\ [g_1|\dots|g_ig_{i+1}|\dots|g_n] \\ [g_1,\dots,g_{n-1}] \end{cases}$$

In low-dimension $C_*(G)$ has the form
$$C_2(G) \xrightarrow{\partial} C_1(G) \xrightarrow{0} \mathbb{Z}$$
where $\partial([g|h]) = [h] - [gh] + [g]$
$\Rightarrow H_0(G) \cong \mathbb{Z}$ and $H_1(G) = G/[G,G]$

Topological interpretation

for $Y \simeq K(G,1)$ the augmented chain complex of the universal cover $\tilde{Y}$ of Y is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

$$C_*(Y) \cong C_*(\tilde{Y}/G) \cong C_*(\tilde{Y})_G \simeq C_*(G)$$

$\hookrightarrow$ as free resolutions are homotopic