BIG PICTURE

Several topological constructions for a space X are determined by the homotopy type of X.

$$\begin{tikzcd} & X \simeq Y \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \\ \pi_n(X) = \pi_n(Y) & H_n(X) = H_n(Y) \arrow[d, Rightarrow] & H^n(X) = H^n(Y) \\ & \chi(X) = \chi(Y) & \end{tikzcd}$$

In 1936 Hurewicz proved the following:

- Prop

  If $\pi_1(X) = \pi_1(Y)$ and $\pi_n(X) = \pi_n(Y) = 0$ $\forall n \geq 2$ then $X \simeq Y$.

If we say $\pi := \pi_1(X)$, then such spaces are "$K(\pi, 1)$s".

Hurewicz's theorem tells us that $K(\pi,1)$ spaces up to homotopy "are" just groups. This gives us an arrow back up the above diagram $\pi_1(X) = \pi_1(Y) \overset{\text{if } K(\pi,1)}{\Longrightarrow} X \simeq Y$.

So, all the constructions that are determined by the homotopy type of a space can be modified to be constructions determined by groups (if we restrict ourselves to $K(\pi,1)$s).

RINGS & MODULES

I include this only because I have had to remind myself about rings & modules while reading through Brown.

- Def: A ring (unital, not necessarily commutative) is an abelian group R equipped with

  i) A unit $1 \in R$
  ii) A bilinear map $\cdot : R \times R \to R$ such that
  $1 \cdot r = r \ \forall r \in R$ & $(r_1 \cdot r_2) \cdot r_3 = r_1 \cdot (r_2 \cdot r_3)$ (associative).

Bilinear means $(r_1 + r_2) \cdot r_3 = r_1 \cdot r_3 + r_2 \cdot r_3$ &
$r_1 \cdot (r_2 + r_3) = r_1 \cdot r_2 + r_1 \cdot r_3$
i.e. a homomorphism $R \otimes R \to R$.

Our rings are not necessarily commutative $\left(r_1 \cdot r_2 \ ?{=}? \ r_2 \cdot r_1 \right)$

I will drop the "$\cdot$". So $rs := r \cdot s$.

For example $\mathbb{Z}$ is a ring, so is $\mathbb{Z}^2$ & $\mathbb{R}$ (although $\mathbb{R}$ is also a field). Polynomial rings are nice.

- Def
  A (left) module over a ring R is an abelian group M & a ring homomorphism

$$\rho : R \longrightarrow \mathrm{End}(M).$$

i.e. $\rho(rs + t) = \rho(r)\rho(s) + \rho(t)$ & $\rho(1) = \mathrm{Id}_M$.

I will write the action of ρ as "$\cdot$" so $r \cdot x := \rho(r)(x)$.
The "left" part comes from $\rho(rs) = \rho(r)\rho(s)$
A right module would have $\rho(rs) = \rho(s)\rho(r)$.

For example $\mathbb{Z}^2$ is a module over $\mathbb{Z}$ (every R^n is a (free) module over R).

A more interesting example is $\mathbb{Z}/2\mathbb{Z}$ is a module over $\mathbb{Z}$ (do all multiplication/addition in $\mathbb{Z}$ then take mod 2).

Def

A module M over R is "free" if $\exists$ $E \subseteq M$ such that E is a generating set for M and E is linearly independent.

i.e. $\forall m \in M$ $\exists$ r_i s.t. $m = \sum_{e_i \in E} r_i \cdot e_i$ &
$\sum_{e_i \in E} r_i \cdot e_i = 0 \Rightarrow r_i = 0 \; \forall i$.

REVIEW OF CHAIN COMPLEXES

Let R be a ring (arbitrary). A graded R-module is a seq. of R-modules $(C_n)_{n\in\mathbb{Z}}$. AA "map of degree p" is a collection of homs $\theta_n : C_n \to C_{n+d}$

A "chain complex" (C,d) over R is a graded R-module with a map d of degree -1 such that $d^2=0$. We usually omit d in our notation "C is a chain complex".

e.g.
$$\begin{tikzcd}
\cdots \mathbb{Z}/4\mathbb{Z} \arrow[r, "\times 2"] \arrow[d, equal] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "\times 2"] \arrow[d, equal] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "\times 2"] \arrow[d, equal] & \mathbb{Z}/4\mathbb{Z} \arrow[r, "\times 2"] \arrow[d, equal] & \cdots \\
C_n & C_{n-1} & C_{n-2} & C_{n-3} &
\end{tikzcd}$$

To such an object we associate the following graded modules.

- Cycles $Z(C) = (Z_n)_{n\in\mathbb{Z}} := (\ker d_n)_{n\in\mathbb{Z}}$
- Boundaries $B(C) = (B_n)_{n\in\mathbb{Z}} := (\operatorname{Im} d_{n+1})_{n\in\mathbb{Z}}$
- Homology $H(C) = (H_n)_{n\in\mathbb{Z}} := (Z_n/B_n)_{n\in\mathbb{Z}}$.

Maybe d has degree $+1$ istead of -1. This is really the same situation as above under a remapping of indices $C_n \leftrightarrow C_{-n}$.
Notationally every "NOUN" becomes "CONOUN" and indices go upstairs. $C_n \rightsquigarrow C^n$.

CHAIN MAPS & HOMOTOPIES

A chain map $f: C \to C'$ is a collection of homs $(f_n : C_n \to C'_n)_{n\in\mathbb{Z}}$. such that all the squares commute.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] \arrow[dr, phantom, "\circlearrowright"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\circlearrowright"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
C'_{n+1} \arrow[r, "d'_{n+1}"'] & C'_n \arrow[r, "d'_n"'] & C'_{n-1}
\end{tikzcd}$$

So $f_n d_{n+1} = d'_{n+1} f_{n+1}$

Suppose there are two chain maps f & g. A homotopy equivalence $f \overset{h}{\simeq} g$ is a collection of homs $(h_n : C_n \to C'_{n+1})_{n \in \mathbb{Z}}$ such that
$d'_{n+1} h_n + h_{n-1} d_n = f_n - g_n$

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, bend right, "f_{n+1}"'] \arrow[d, bend left, "g_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[dl, "h_n"'] \arrow[d, bend right, "f_n"'] \arrow[d, bend left, "g_n"] & C_{n-1} \arrow[dl, "h_{n-1}"'] \arrow[d, bend right, "f_{n-1}"'] \arrow[d, bend left, "g_{n-1}"] \\
C'_{n+1} \arrow[r, "d'_{n+1}"] & C'_n \arrow[r, "d'_n"] & C'_{n-1}
\end{tikzcd}$$

If $\exists\, h$ $f \overset{h}{\simeq} g$ then we say $f \simeq g$.

- Prop:
  A chain map $f : C \to C'$ induces a map $H(f) : H(C) \to H(C')$ (Usually f_*)
  and $H(f)$ is an isomorphism if $f \simeq g$.

  We say $f : C \to C'$ is a homotopy equiv if $\exists\, f' : C' \to C$
  s.t. $f'f \simeq id_C$ & $ff' \simeq id_{C'}$.

- Prop:
  If $\exists$ a homotopy equiv $f : C \to C$ then $H(f)$ is an isomorphism.

HOM POINT OF VIEW

Given two chain complexes C & C', we define a "function complex" $Hom(C, C')$.
The graded modules are all the degree n homs $C \to C'$, so

$$Hom(C, C')_n = \prod_{q \in \mathbb{Z}} Hom(C_q, C'_{q+n}).$$

The boundary operator $D_n : Hom(C, C')_n \to Hom(C, C')_{n-1}$ is defined as

$$D_n(f) = d'f - (-1)^n f d$$

$$\begin{tikzcd}
C_q \arrow[r, "d_q"] \arrow[dr, "f_q"'] \arrow[drr, dashed, bend left, "D_{-1}(f)_q"] & C_{q-1} \arrow[r, "d_{q-1}"] \arrow[dr, "f_{q-1}"] & C_{q-2} \\
C'_q \arrow[r, "d'_q"] & C'_{q-1} \arrow[r, "d'_{q-1}"] & C'_{q-2}
\end{tikzcd}$$

$f \in Hom(C, C')_{-1}$

$$D_{-1}(f)_q = d'_{q-1} f_q - (-1)^{-1} f_{q-1} d_q$$

$\cap$

$Hom(C, C')_{-2}$

So $D^2 = 0$.

0-cycles are $Ker(D_0)$

$$f \in Ker(h \mapsto d'h - hd) \iff d'f = fd \qquad = -(-1)^0$$

(Chain map commuting square condition)

So 0-cycles are the chain maps $C \to C'$.

0-Boundaries are $Im(D_1)$

$$f \in Im(h \longmapsto d'h + hd) \iff \exists h \quad f = f - 0 = d'h + hd \qquad = -(-1)^1$$

$\iff f \overset{h}{\simeq} 0.$

So 0-boundaries are the null homotopic chain maps.

Now consider $H_0(Hom(C, C'))$. This is $\dfrac{Ker(D_0)}{Im(D_1)} = \dfrac{\text{Chain maps } C \to C'}{\text{Null hom. } C \to C'}$

which is exactly "Chain maps $C \to C'$ up to homotopy". We use $[C, C']$ as notation for this.

SNAKE LEMMA

- Prop:

  A short exact seq. of chain complexes gives a long exact sequence in homology (snake lemma).

i.e. given $0 \to C' \xrightarrow{i} C \xrightarrow{\pi} C'' \to 0$ exact with i, π chain maps. There exists a L.E.S.

$$\cdots H_n(C') \xrightarrow{H(i)} H_n(C) \xrightarrow{H(\pi)} H_n(C'') \xrightarrow{\partial} H_{n-1}(C') \longrightarrow \cdots$$

MAPPING CONE

Assume $H_*(C'')=0$. Then by the above, we would have

$$\underset{\overset{\parallel}{0}}{H_n(C'')} \longrightarrow H_n(C') \longrightarrow H_n(C) \longrightarrow \underset{\overset{\parallel}{0}}{H_{n-1}(C'')} \quad \Rightarrow$$

$$H_n(C') \cong H_n(C)$$

So, $H_*(C'')$ can describe how $H_*(C')$ fails to be isomorphic to $H_*(C)$.

The "cokernel" of a chain map $f: C \to C'$ is

$C/f(C)$, and we require this to be a chain complex.
On each module we have the exact seq.

$$0 \longrightarrow C_n \xrightarrow{f_n} C'_n \xrightarrow{\pi_n} \mathrm{Coker}(f_n) \longrightarrow 0$$

Now we want the following diagram, which would show that $\mathrm{Coker}(f)$ is a chain complex and $\pi: C \to \mathrm{Coker}(f)$ is a chain map. Let $C''_q := \mathrm{Coker}(f_q)\ \forall q$.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] \arrow[dr, phantom, "\circlearrowleft"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\circlearrowleft"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
C'_{n+1} \arrow[r, "d'_{n+1}"] \arrow[d, "\pi_{n+1}"'] \arrow[dr, phantom, "\circlearrowleft"] & C'_n \arrow[r, "d'_n"] \arrow[d, "\pi_n"] \arrow[dr, phantom, "\circlearrowleft"] & C'_{n-1} \arrow[d, "\pi_{n-1}"] \\
C''_{n+1} \arrow[r, dashed, "d''_{n+1}", "\uparrow \text{want}"'] & C''_n \arrow[r, dashed, "d''_n", "\uparrow \text{want.}"'] & C''_{n-1}
\end{tikzcd}$$

By the universal property of the quotient, we can factor $\pi_n d'_{n+1}$ uniquely through π_{n+1} iff

$$\ker(\pi_{n+1}) = \mathrm{Im}(f_{n+1}) \subseteq \ker(\pi_n d'_{n+1}) \; \left(\iff \pi_n d_{n+1} f_{n+1} = 0\right).$$

By the above commuting square we have

$$(\pi_n d'_{n+1}) f_{n+1} = \pi_n (d'_{n+1} f_{n+1}) = \pi_n (f_n d_{n+1}) = \underbrace{(\pi_n f_n)}_{0} d_{n+1} = 0.$$

So yes, $\mathrm{Im}(i_{n+1}) \subseteq \ker(\pi_n d_{n+1})$

So if we start off with an inclusion chain map $i : C' \hookrightarrow C$, then we can cook up a S.E.S.

$$0 \longrightarrow C' \overset{i}{\hookrightarrow} C \longrightarrow \mathrm{Coker}(i) \longrightarrow 0$$

and get a corresponding L.E.S. using the snake lemma.
But, what if we just have a generic chain map $f : C' \longrightarrow C$ (not injective). In this case we want to construct a chain cx A & B (using $f : C' \longrightarrow C$) s.t. we have a S.E.S.

$$0 \longrightarrow C \longrightarrow A \longrightarrow B \longrightarrow 0.$$

We do this using the "mapping cone of $f : C' \longrightarrow C$."
To define this, first define the "suspension of C", ΣC to be

$$(\Sigma C)_n = C_{n-1} \quad \& \quad \Sigma d = -d$$

The mapping cone is defined to be the chain cx (C'', d'') where

$$C'' := C \oplus \Sigma C' \qquad \text{and} \qquad d''(c, c') = (dc + fc', -d'c')$$

i.e. (recall $f : C' \longrightarrow C$)

$$\cdots \begin{matrix} C_n \\ \oplus \\ C'_{n-1} \end{matrix} \xrightarrow{d''} \begin{matrix} C_{n-1} \\ \oplus \\ C'_{n-2} \end{matrix} \xrightarrow{d''} \begin{matrix} C_{n-2} \\ \oplus \\ C'_{n-3} \end{matrix} \cdots \qquad d'' = \begin{pmatrix} d & f \\ 0 & \Sigma d' \end{pmatrix}$$

Explicitly: $d''(\overset{C_n}{c}, \overset{C'_{n-1}}{c'}) = (d_n c + f_{n-1} c', -d'_{n-1} c')$

We see $(d'')^2 = 0$ by squaring the matrix.

$$\begin{pmatrix} d & f \\ 0 & \Sigma d' \end{pmatrix}^2 = \begin{pmatrix} d^2 & df + f\Sigma d' \\ 0 & (\Sigma d')^2 \end{pmatrix} = 0$$

since $f\Sigma d' = -fd'$ & f is a chain map.

We have a S.E.S. $0 \to C \longrightarrow C \oplus \Sigma C' \longrightarrow \Sigma C' \to 0$ and when we apply the connecting homomorphism from the snake lemma we find $\partial : H_n(\Sigma C') \to H_{n-1}(C)$ is exactly $\underset{(f_*)_{n-1}}{\overset{\|}{H(f)_{n-1}}} : C'_{n-1} \to C_n$.

So, the L.E.S.

$$\cdots H_n(C') \xrightarrow{H(f)_n} H_n(C) \longrightarrow H_n(C \oplus \Sigma C') \longrightarrow H_{n-1}(C') \longrightarrow \cdots$$

tells us that $H(f)$ is an isomorphism (f is a weak equivalence) iff $H(C \oplus \Sigma C') = 0$ ($C \oplus \Sigma C'$ is acyclic).

- Prop:
  $f : C \longrightarrow C'$ is a homotopy equiv. iff its mapping cone $C \oplus \Sigma C'$ is contractible.

Lastly, we just state the universal coeff thm.

- Thm:
  R a PID & M an R-module. The following are exact:

$$0 \to H_n(C) \otimes_R M \to H(C \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(C), M) \to 0$$

&

$$0 \longrightarrow \mathrm{Ext}_R^1(H_{n-1}(C), M) \to H^n(\mathrm{Hom}_R(C, M)) \to \mathrm{Hom}_R(H_n(C), M) \to 0$$

FREE RESOLUTIONS

R a ring with identity, M a (left) R-Module.

- Def

  A "resolution" of M is an exact seq. of R-modules.

  $$\cdots \longrightarrow F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

  If each F_i is free, then this is a "free resolution".

Free resolutions can always be cooked up:

1) Choose a surjection $F_0 \overset{\varepsilon}{\twoheadrightarrow} M$ (Choose a gen. set for M).
2) Choose a surjection $F_1 \twoheadrightarrow \ker(\varepsilon)$ (Choose relations for M).
3) ...

If $\exists$ $0 \to F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ then we say the resolution has length $\leq n$.

- Examples
  - A free module F has the free resolution $0 \to F \to F \to 0$.
  - If $R = \mathbb{Z}$ or any PID, then submodules of free modules are free (NOT OBVIOUS). So $\ker(\varepsilon) \leq F_0$ is free, giving us the following free resolution.

  $$0 \longrightarrow \ker(\varepsilon) \longrightarrow F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

  e.g. $0 \longrightarrow 2\mathbb{Z} \hookrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$ or

  $0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$

  - Let $R = \dfrac{\mathbb{Z}[T]}{(T^2-1)}$ with t image of T in R $\left(t = T + (T^2-1)\right)$. Let M be the R-module $R/(t-1)$

  We have the defining projection $R \xrightarrow{\pi} M$

And since $(T^2-1)=(T-1)(T+1)$, something is in the nontrivial kernel of π (multiples of $(t-1)$) iff it is a multiple of $(t+1)$. So the following is exact.

$$R \xrightarrow{(t-1)} R \xrightarrow{\pi} M \longrightarrow 0$$

The non trivial kernel of $R \xrightarrow{(t-1)} R$ is exactly all the multiples of $(t+1)$ and so on... so the following is exact.

$$\cdots R \xrightarrow{(t-1)} R \xrightarrow{(t+1)} R \xrightarrow{(t-1)} M \longrightarrow 0.$$

And R is a free module with basis $\{1\}$, so this is a free projection.

N.B. If M is an R-module, then F_i need to be free <u>over R</u>.

GROUP RINGS

G a group, then $\mathbb{Z}G$ (or $\mathbb{Z}[G]$) be the free $\mathbb{Z}$-module generated by the elements of G. i.e.

$$\mathbb{Z}G = \left\{ \sum_{g\in G} a(g)\, g \;\middle|\; a(g)=0 \text{ almost everywhere} \right\}.$$

Multiplication in G extends linearly to $\mathbb{Z}G \times \mathbb{Z}G \longrightarrow \mathbb{Z}G$ making $\mathbb{Z}G$ a ring. The "integral group ring" of G.

Suppose we have a ring hom. $\mathbb{Z}G \xrightarrow{f} R$. This is defined by the mapping of the generators $G \subseteq \mathbb{Z}G$, i.e. we need to specify $f(g)$ $\forall g \in G$. If $r = f(g)$ then $f(g)^{-1} r = f(g)^{-1} f(g) = f(g^{-1}g) = 1$ so $r \in R^\times$, and f defines a group hom $G \to R^\times$.
Also a group hom $G \to R^\times$ extends uniquely to a ring hom $\mathbb{Z}G \to R$ so.

- Prop

$$\mathrm{Hom}_{\mathrm{ring}}(\mathbb{Z}G, R) \approx \mathrm{Hom}_{\mathrm{group}}(G, R^\times).$$

- Example
  Let G = cyclic group of order n. Then $\exists\, t \in G$ with $\{t^i\}_{1 \le i \le n} = G$ forms a basis for $\mathbb{Z}G$.
  So $\mathbb{Z}G \approx \frac{\mathbb{Z}[T]}{T^n - 1}$.

G MODULES

Recall $\mathbb{Z}G$ has a ring structure.

- Def
  A (left) $\mathbb{Z}G$-Module (also called a G-Module) is a (left) module over $\mathbb{Z}G$.
  That is... an abelian group A along with a ring homomorphism $\rho: \mathbb{Z}G \to \mathrm{End}(A)$.

In light of $\mathrm{Hom}_{ring}(\mathbb{Z}G, R) \approx \mathrm{Hom}_{group}(G, R^{\times})$, this means

$$\mathrm{Hom}_{ring}(\mathbb{Z}G, \mathrm{End}(A)) \approx \mathrm{Hom}_{group}(G, \mathrm{End}(A)^{\times}) = \mathrm{Hom}_{group}(G, \mathrm{Aut}(A)).$$

So, G-modules are just an abelian group A along with a G-action on A.

We can cook up a G-Module from any G-Set X.
Let X be a G-Set. Let $\mathbb{Z}X$ denote the free abelian group generated by X. We can extend $G \curvearrowright X$ to a $\mathbb{Z}$-linear action $G \curvearrowright \mathbb{Z}X$ $(g \cdot (x_1 + x_2) = g \cdot x_1 + g \cdot x_2)$.
The resulting G-module is called a "permutation module".

For example G acts on its set of left cosets for any $H \le G$. $(G \curvearrowright G/H)$.

We can combine two G-sets by taking the disjoint union. if $G \curvearrowright X_1$ & $G \curvearrowright X_2$ then $G \curvearrowright (X_1 \sqcup X_2) =: X$ where the action $g \cdot x$ depends on whether $x \in X_1$ or $x \in X_2$.

The resulting permutation module is the direct sum

$$\mathbb{Z}[X_1 \sqcup X_2] = \mathbb{Z}X_1 \oplus \mathbb{Z}X_2$$

Every G-set X decomposes in this way in to its respective orbits. Furthermore $G \curvearrowright G\,\mathrm{Stab}(x) \underset{G\text{-sets}}{\cong} G \curvearrowright \mathrm{Orb}(x)$

$$\begin{tikzcd}
g\,\mathrm{Stab}(x) \arrow[r, mapsto, "\cdot h"] \arrow[d, mapsto, "s"'] \arrow[dr, phantom, "\circlearrowright"] & hg\,\mathrm{Stab}(x) \arrow[d, mapsto, "s"] \\
g \cdot x \arrow[r, mapsto, "\cdot h"'] & hg \cdot x
\end{tikzcd}$$

So, $\mathbb{Z}X \approx \bigoplus_{x \in E} \mathbb{Z}[G/\mathrm{Stab}(x)]$

where $E \subseteq X$ is a set of representatives for X/G.

So, if $G \curvearrowright X$ is free ($\mathrm{Stab}(x) = 1\ \forall x \in X$), then $\mathbb{Z}X$ is a free $\mathbb{Z}G$ module with basis E.

RESOLUTIONS OF $\mathbb{Z}$ OVER $\mathbb{Z}G$ VIA TOPOLOGY

We always have the trivial action of G on an abelian group $g \cdot a = a$ $\forall g$ $\forall a$. So, here we consider $\mathbb{Z}$ as a G-module with trivial action. We will then show that free resolutions of this module can arise from free actions of G on complexes $\simeq *$.

Def
A G-complex is a CW-complex with a cellular G-action. i.e. $x \mapsto g \cdot x$ is a homeomorphism that preserves cells.

Recall the cellular chain complex for a CW-cx X.

$$C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \to 0.$$

Where each $C_n \cong \mathbb{Z}^q$ where $q = \#\{n\text{-cells in } X\}$.
The boundary maps relate to the degree of the attaching maps of the n-cells to the $n-1$ skeleton X^{n-1}.

Moreover, if X is a G-Complex, then G permutes n-dimensional cells, so each C_n is a G-module. The "augmentation map" $\varepsilon : C_0 \to \mathbb{Z}$ where $\varepsilon(v) = 1$ $\forall$ 0-cells v, is also a G-module map. So... we have a chain complex of G-modules.

To see that the boundary map is indeed a G-Module map, we need to show that it is G-equivariant. i.e.
$g \cdot \partial(e_a) = \partial(g \cdot e_a)$ $\forall$ n-cells e_a.

Since X is a G-complex, the attaching map of the cell $g \cdot e_a$ is the map $g \circ \varphi_a$ where $\varphi_a : S^{n-1} \to X^{n-1}$ is the attaching map of the cell e_a.

$$\begin{tikzcd} S^{n-1} \arrow[r, "\varphi_e"] \arrow[d, "id"'] & \bigvee S^{n-1} \arrow[d, "S\ g"] \\ S^{n-1} \arrow[r, "\varphi_{g\cdot e}"] & \bigvee S^{n-1} \end{tikzcd}$$

i.e. we have this commutative diagram.

So $\underbrace{\deg\left(\partial(e_a) \xrightarrow{\pi\circ\varphi_a} \bigvee S^{n-1} / \bigvee S^{n-1}\setminus e_b\right)}_{\chi^{ab}} = \underbrace{\deg\left(\partial(g\cdot e_a) \xrightarrow{\pi\circ\varphi_{g\cdot a}} \bigvee S^{n-1} / \bigvee S^{n-1}\setminus g\cdot e_b\right)}_{\chi^{g\cdot a\ g\cdot b}}$

We have $g\cdot\partial(e_a) = g\cdot\sum \chi^{ab} e_b = \sum \chi^{ab} e_{g\cdot b}$

& $\partial(g\cdot e_a) = \partial(e_{g\cdot a}) = \sum_b \chi^{g\cdot a\ b} e_b$

But $b \mapsto g\cdot b$ is just a permutation, so

$$\sum_b \chi^{g\cdot a\ b} e_b = \sum_b \chi^{g\cdot a\ g\cdot b} e_{g\cdot b} \qquad \text{By compatibility}$$

$$= \sum_b \chi^{ab} e_{g\cdot b} = g\cdot\partial(e_a).$$

So... Indeed the cellular chain cx is a chain cx of G-modules. Furthermore if the G-complex is "free", i.e. permutes n-cells freely, then each of these $\tilde{C}_n$ is a <u>free</u> G-module. <u>And</u> the homology of the cellular chain complex is the homology of X, so if $H_n(X)=0\ \forall n\geq 2$ then the cellular chain cx. we have made is exact. So

- Prop:
  Let X be a contractible free G-complex then the augmented cellular chain complex (with $\varepsilon: C_0 \to \mathbb{Z}$ as above) is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

This ties in nicely with regular covers.

- Def
  A covering map $p: \tilde{X} \to X$ with group of deck transformations G is said to be "regular" if

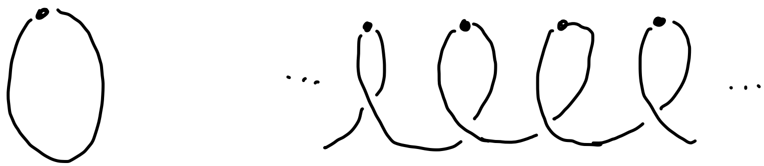
  i) G acts transitively on $p^{-1}(x)$ $\forall x$ (i.e. $X \cong \tilde{X}/G$).
  ii) The image of $\pi_1(\tilde{X}) \xrightarrow{p_*} \pi_1(X)$ is normal in $\pi_1(X)$ (for one and hence all choices of basepoints).
  iii) $\forall$ closed loops ω in X, if $\exists$ a lift of ω which is closed then all lifts of ω are closed.

- Prop
  Given a regular cover $p: \tilde{X} \to X$ with group of deck transformations G, we have
  $$G \cong \pi_1(X) / p_*(\pi_1(\tilde{X}))$$

So if $\tilde{X}$ is simply connected ($\tilde{X}$ is universal cover of X) then $G \cong \pi_1(X)$.

- Prop
  Suppose $p: \tilde{Y} \to Y$ is a regular cover with $G \curvearrowright \tilde{Y}$. ($Y \cong \tilde{Y}/G$)
  If Y is a CW-cx then $\tilde{Y}$ inherits a G-complex structure
  In this construction, the action of G on each $p^{-1}(\sigma)$ ($\sigma \subseteq Y$ an open cell) is free (on cells).

e.g. $Y \xleftarrow{p} \mathbb{Z} \curvearrowright \tilde{Y}$ $\qquad$ $\mathbb{Z}$ free on 0-cells & 1-cells in $\tilde{Y}$.



So, if we have a CW-complex Y such that:

i) Y is connected

ii) $\pi_1(Y) = G$

iii) Universal cover of Y is contractible

• Def:

Such a Y is called a $K(G,1)$-complex or "Eilenberg MacLane complex".

Then $G \curvearrowright \tilde{Y}$ freely and this gives us a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. The modules C_n are free products of #{orbits of n-cells}.

• Prop

Condition (iii) can be replaced by

iii') $H_i Y = 0$ for $i \geq 2$ $\qquad$ (CW-complex only)

iii'') $\pi_i Y = 0$ for $i \geq 2$