Enum Geo Reading group

06/10/2025

- Mondays, 11 - 1pm
- Hybrid
- Bayes 1.35
- Credits ⟹ have to present
- Lecture notes uploaded to website

Enum Geo (Ricolfi)

Motivation: String theory

We want enumerate geometric objects w/ properties:

- lines on a cubic surface
- stables maps onto a target
- torsion free sheaves on CY3

"How many lines go through one point in $\mathbb{C}^2$?"

"How many Weierstrass points are there on a given genus g curve $/\mathbb{C}$?"

What is the right question to ask?

Recipe:

A) Construct a moduli space M.

B) Compactify M.

C) impose $\dim M$-many conditions to expect a finite answer.

D) Count via intersection theory on M.

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c) each condition (go through points, be tangent conics, etc.) is described by a cycle $Z_i \subset M$.

$PD(Z_i) = \alpha_i \in A^* M$.

General condition $\alpha_1, \dots, \alpha_r$

$$\alpha = \alpha_1 \cup \dots \cup \alpha_r$$

We pick conditions Z_i s.t.:

$$\sum_{i=1}^{r} \operatorname{codim}(Z_i, M) = \dim M.$$

$\Longrightarrow$ right question?

D) $\int_M \alpha = \deg_M (\alpha \cap [M]) \in \mathbb{Z}$

$\deg_M : A_*M \to A_* pt = \mathbb{Z}$

How many lines are on a smooth cubic surface $S \subset \mathbb{P}^3$?

A) Set up M.

lines on $\mathbb{P}^3 \cong Gr(2,4)$

B) compactify M.

Why is $Gr(2,4)$ compact?

It's projective.

Embed $Gr(2,4)$ into $\mathbb{P}^N$,

$$i : Gr(2,4) \hookrightarrow \underline{\mathbb{P}(\Lambda^2 V)} \cong \mathbb{P}^5$$

$\underset{2d}{W} \subset \underset{4d}{V}$, let $W = \mathrm{span}(v, w)$, we

send W to the $\underline{\text{point}}$ $[v \wedge w]$.

If we take basis V $e_0, \dots, e_3$

$v = (v_i)$, $w = (w_i)$

$$p_{ij} = v_i w_j - v_j w_i \quad 0 \le i < j \le 3$$

$$p_{01} p_{23} - p_{02} p_{13} + p_{03} p_{12} = 0$$

$\subset \mathbb{P}^5$

$\Rightarrow$ $Gr(2,4)$ has to be compact.

A) and B) checked!

Problem is now about $Gr(2,4)$

$$\underline{\Sigma_{0,0}} = \left(\mathbb{1} \;\; \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix} \right), \quad \underline{\Sigma_{1,0}} = \begin{pmatrix} 1 & \bullet & 0 & \bullet \\ 0 & 0 & 1 & \bullet \end{pmatrix}$$

$$\Sigma_{2,0} = \begin{pmatrix} 1 & \bullet & \bullet & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \underline{\Sigma_{1,1}} = \begin{pmatrix} 0 & 1 & 0 & \bullet \\ 0 & 0 & 1 & \bullet \end{pmatrix}$$

$$\Sigma_{2,1} = \begin{pmatrix} 0 & 1 & \bullet & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \underline{\Sigma_{2,2}} = \left(\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \;\; \underline{\mathbb{1}} \right)$$

where $\bullet$ is a complex number.

$$\left\{ \underline{\{\}} \subset \underline{\square} \subset \begin{matrix} \square\square \\ \begin{smallmatrix}\square\\\square\end{smallmatrix} \end{matrix} \subset \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \subset \begin{smallmatrix}\square\square\\\square\square\end{smallmatrix} \right\}$$

$$\Sigma_{2,2} \subset \Sigma_{2,1} \subset \Sigma_{1,1} \cup \Sigma_{2,0} \subset \Sigma_{1,0} \subset \Sigma_{0,0}$$

$\longrightarrow$ Are closed, as well as complex

$$H^0(Gr(2,4), \mathbb{Z}) = \mathbb{Z} b_{0,0}, \quad H^2 = \mathbb{Z} \underline{b_{1,0}}$$

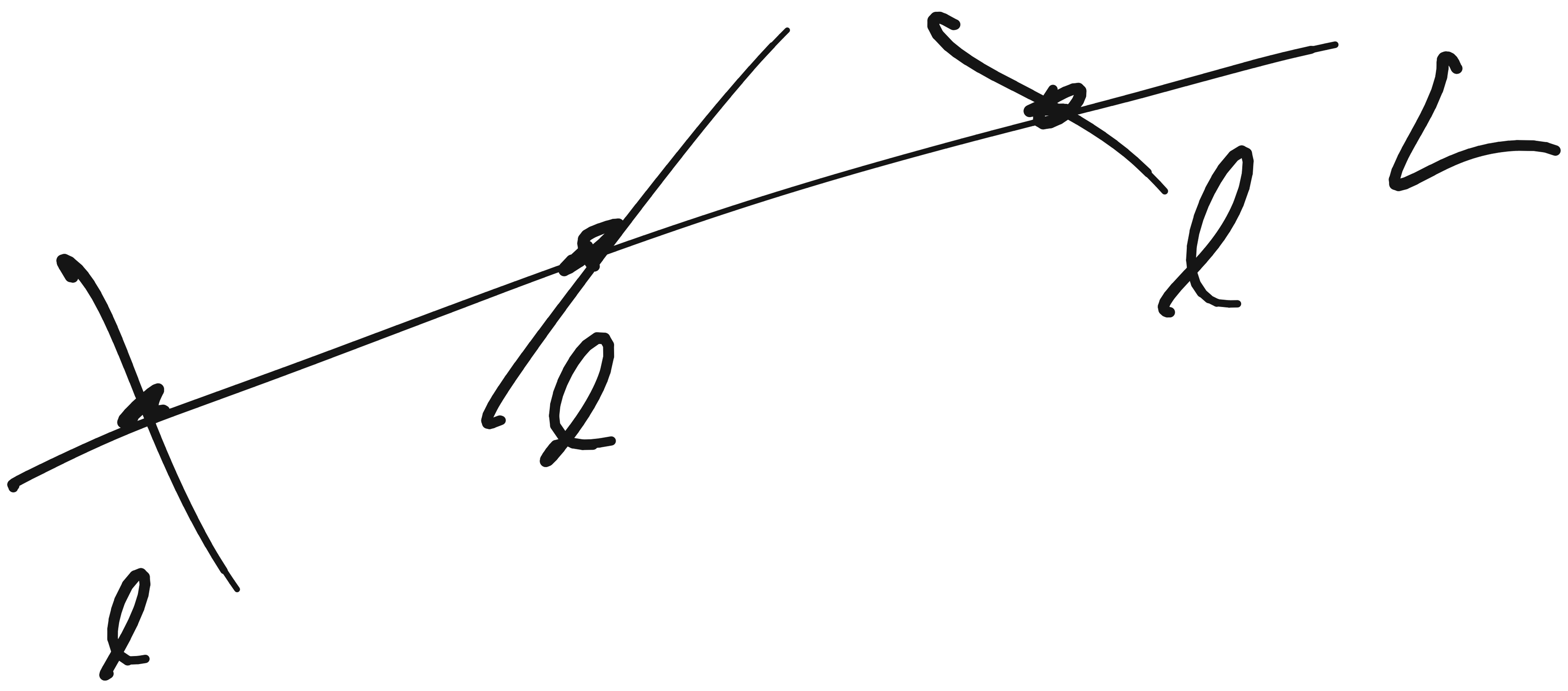
$$\underline{H^4 = \mathbb{Z} b_{2,0} \oplus \mathbb{Z} b_{1,1}}, \quad H^6 = \mathbb{Z} \underline{b_{2,1}}$$

$$H^8 = \mathbb{Z} b_{2,2}.$$

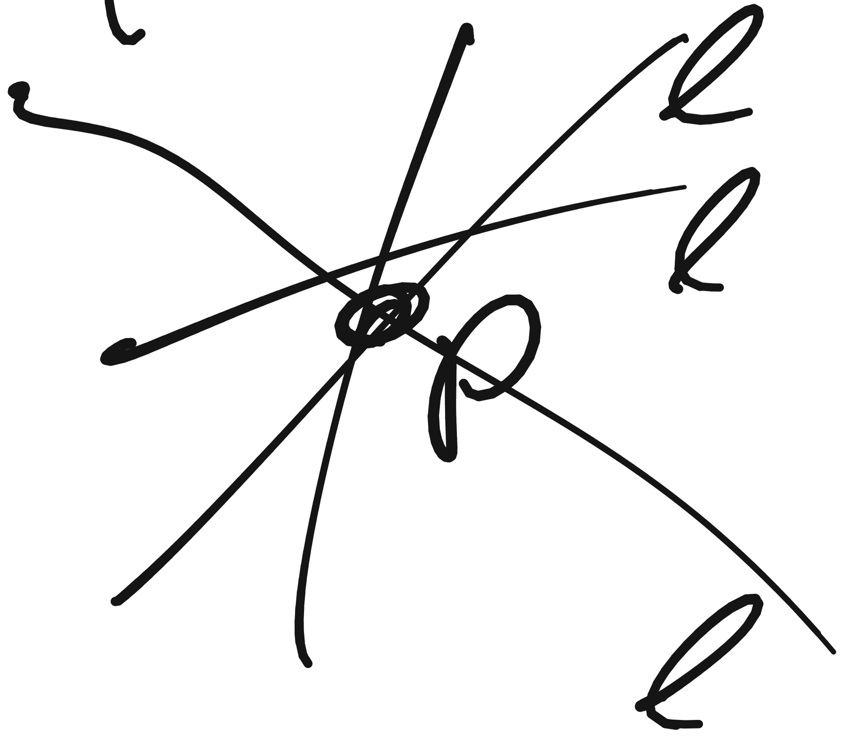
Fix a flag $p \in L \subset H$, $p = [1:0:0:0]$

$L = \{[\bullet:\bullet:0:0]\}$, $H = \{[\bullet:\bullet:\bullet:0]\}$

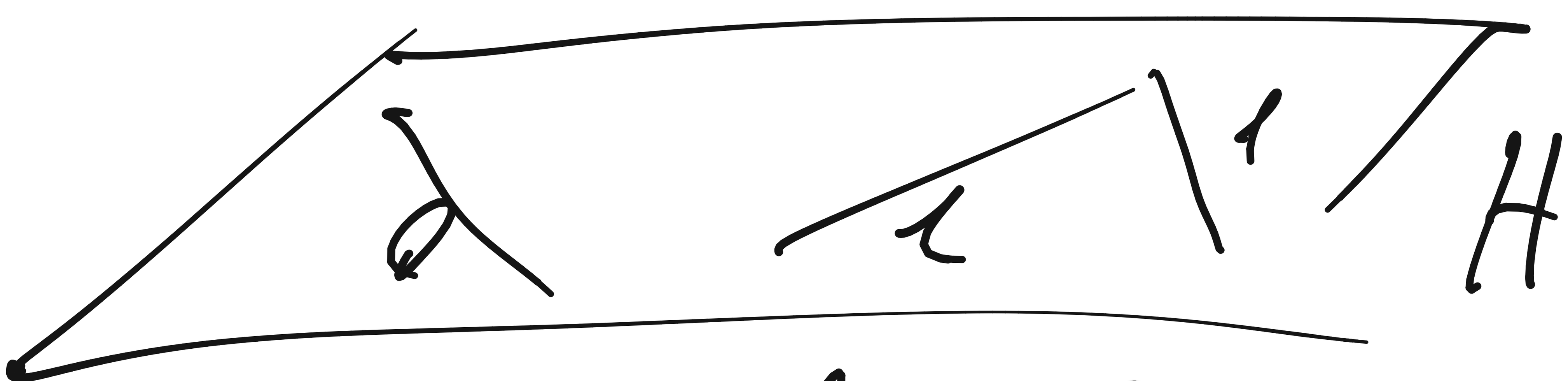
$\sigma_{1,0} = \{\ell \mid \ell \cap L \neq \emptyset\}$



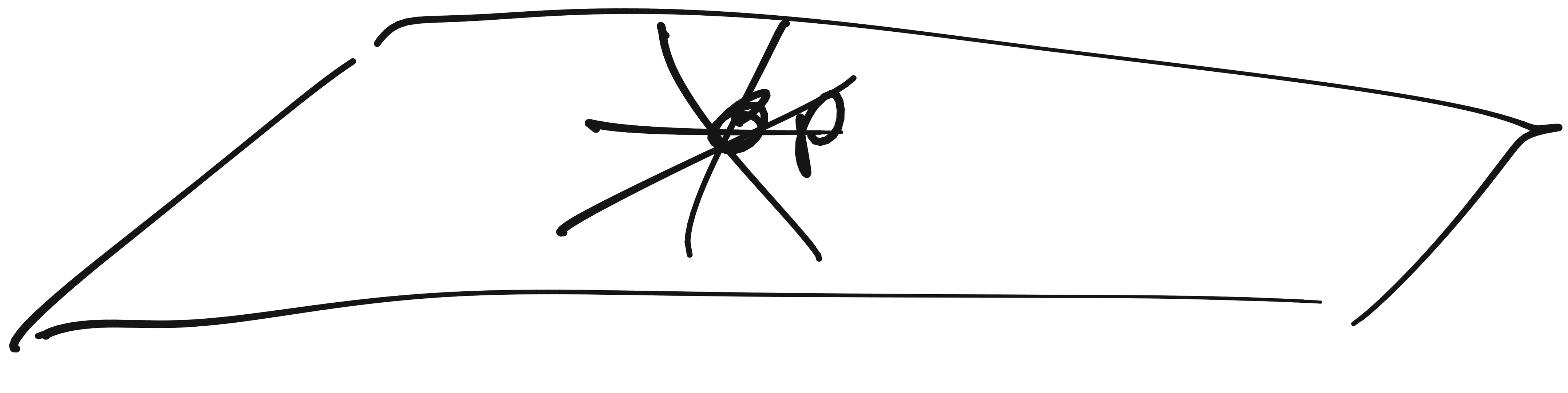
$\sigma_{2,0} = \{\ell \mid p \in \ell\}$



$\sigma_{1,1} = \{\ell \mid \ell \subset H\}$



$\sigma_{2,1} = \{\ell \mid p \in \ell \subset H\}$



$\boxed{\sigma_{11} \cdot \sigma_{20} = ?}$ (Fix a new flag $p' \in L' \subset H'$)

$\sigma_{11} \cdot \sigma_{20}$ should be line $l \subset H$ which meet p' $\Rightarrow$ $p' \notin H$ generically so

$$\sigma_{11} \cdot \sigma_{20} = 0$$

$\sigma_{11}^2 = ?$ $l \subset H$ and $l \subset H'$

but $H \cap H'$ intersect in one line $\hat{L}$

generically!

$$\sigma_{11}^2 = \{l \mid l \in \hat{L}\} = \sigma_{22}$$

$\{F = 0\} = S$, $l \in \mathbb{P}^3$, now restrict $F|_l$ which will be a zero section of a bundle: ($E \to Gr(2,4)$ Tautological bundle)

$$V = Sym^3 E^*$$

[Fiber of V over a line is the space homogeneous cubic polynomials on l.]

$S = 0$ is the set of lines on S.

$\Rightarrow$ What is

$$\int_{Gr(2,4)} c_{Top}(V) = ?$$

$$c_4(\mathrm{Sym}^3 E^*) = 9\, c_2(E^*)\left(2c_1(E^*)^2 + c_2(E^*)\right)$$

$c_1(E^*) = b_{1,0}$, $c_2(E^*) = b_{1,1}$

$$c_4(\mathrm{Sym}^3 E^*) = 9\, b_{1,1}\underbrace{\left(2 \cdot b_{1,0}^2 + b_{1,1}\right)}_{b_{2,2}} =$$

$$= 27\, b_{2,2}$$

$$\int_{Gr(2,4)} c_4(\mathrm{Sym}^3 E^\vee) = \int_{Gr(2,4)} 27\, b_{2,2} =$$

$$\boxed{= 27}$$

Conclusion to the 27 lines:

Very ad-hoc $\Rightarrow$ Torus localization.

How many lines on a quintic threefold?

$\Rightarrow$ 2875.

$$(\mathring{C}, p_1, \ldots, p_{n-1}) \longrightarrow X$$

$$\overline{\mathcal{M}}_{g,n}$$

$$P_{n,\beta}(X) \quad , \quad (F, s)$$