Category of Constructible Sheaves & Local Systems

Sheaves in Topology - Dimca '04
Perverse sheaves ... - Achar '21

Plan
- Stratifications & constructible sheaves
- (Perverse) t-structures
  - Introduce t-structures (A.7)
  - the perverse t-structure (3.1)
- Local Systems ⟺ reps of $\pi_1(X)$
  (paper by Simpson)
  (Narasimhan-Seshadri correspondence $k=\mathbb{C}$ or NAHC

1. Constructible Sheaves

X scheme $/R$ (R - comm ring)
(i.e. $\mathrm{Spec}(A) \subseteq X$, A R-CAlg.)

Def A sheaf $\mathcal{F}$ on X of R-mod (sometimes Ab) is local system if $\exists (U_\alpha)_{\alpha \in I}$ open covering s.t. $\mathcal{F}_{U_\alpha}$ is constant

Facts (1) $R = K$ field
$\Rightarrow$ equivalent to K-lin rep of $\pi_1(X)$
(X path connected)

(2) X K-variety (fin type + separated)
loc sys $\Rightarrow$ coherent $\mathcal{O}_X$-Mod. & locally free
$\Updownarrow$ RH-corr.
($\mathcal{D}$-modules (regular holonomic))

(3) $f : X \to Y$ morph $/R$

f^* maps locsys to locsys

Def X conn. $\mathcal{F}$ loc sys, $\mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\underbrace{\mathcal{F}_x}_{\text{stalk}})$

$$\mathrm{grade}(\mathcal{F}) = \mathrm{grade}(\mathcal{F}_x) = \mathrm{depth}_R(\mathrm{Ann}(\mathcal{F}_x))$$
$$= \min\{ i \mid \mathrm{Ext}^i(R, \mathrm{Ann}(\mathcal{F}_x)) \}$$

(all valued in R-Mod)

$$\mathrm{Loc}^{ft}(X) \underset{\text{full}}{\subseteq} \mathrm{Loc}(X) \underset{\text{full}}{\subseteq} \mathrm{Sh}(X)$$

Idea Stratify $X = \bigsqcup_{i \in I} X_i$ R-variety (sep. f.t. $/R$)

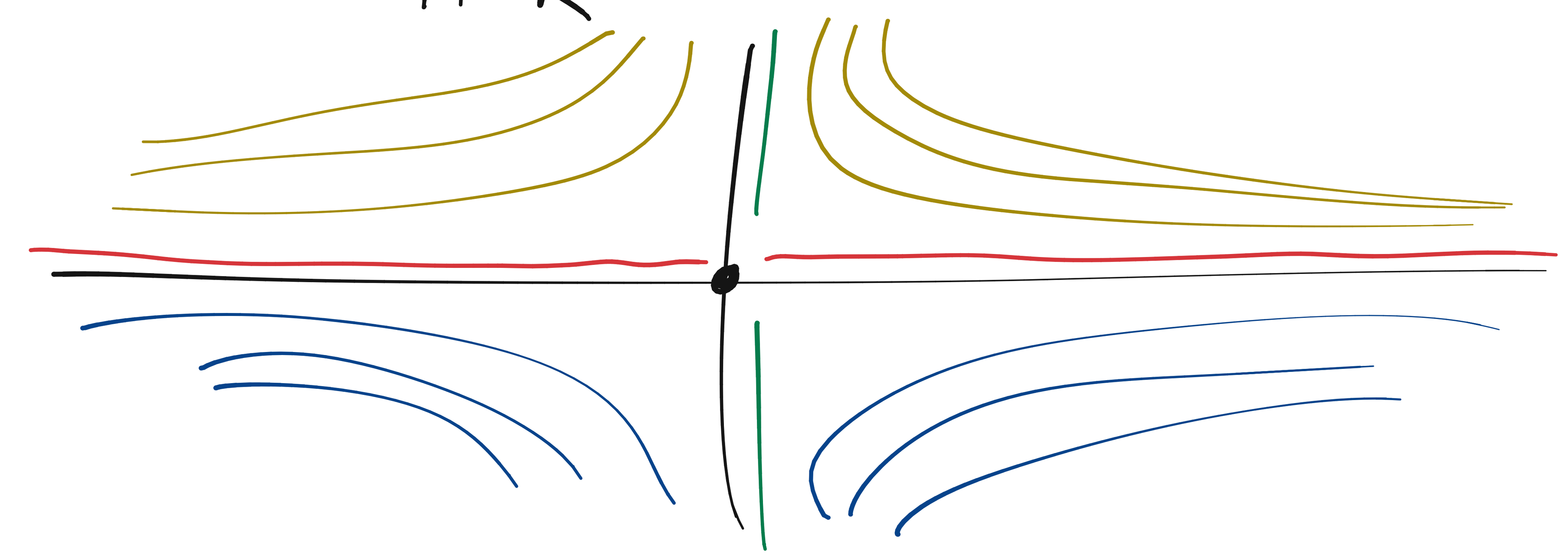
$\mathcal{F}_{X_i}$ is a local system (of finite type)

Def $X = \bigsqcup_{i \in I} X_i$ is stratification $\mathcal{I}$ if

$\forall i,j \in I : \overline{X_i} \cap X_j = \emptyset, X_j$

Order: $i \le j \Leftrightarrow X_i \subseteq \overline{X_j}$

E.g. $R^\times \curvearrowright \mathbb{A}^2_k$ $\quad t(x,y) = (tx, t^{-1}y)$



Fix strat $\mathcal{I}$.

Def $\mathcal{F} \in \mathrm{Sh}(X)$ constructible (wrt. $\mathcal{I}$) if $\mathcal{F}_{X_i}$ loc sys of finite type

$$\mathrm{Sh}_{\mathcal{I}}(X) \subseteq \mathrm{Sh}_{constr}(X) \subseteq \mathrm{Sh}(X)$$

Facts (1) $\oplus$, $\otimes$, Ker, Im, Coker

Closed under constr.

(2) Closed under extensions

Def $\mathcal{F} \in D^b(X)$ constructible if $H^k(\mathcal{F})$ constr $\forall k$.

$$\rightsquigarrow D^b_{\mathcal{I}}(X) \subseteq D^b_{constr}(X) \subseteq D^b(X)$$

(Same for local sys. $D^b_{locsys}(X)$)

Thm (1) $\otimes$, $RHom(-,-)$ closed

as a sheaf

$f: X \longrightarrow Y$ morph $/ K$

$\mathcal{F} \in D^b_{constr}(X)$, $\mathcal{G} \in D^b_{constr}(Y)$

(2) $f^{-1}\mathcal{G}$, $f^!\mathcal{G}$ are constr

(3) $Rf_* \mathcal{F}$ and $Rf_! \mathcal{F}$ are constr

proof sketch of (1)

constr. closed under distinguished triangles & shifts

$\rightsquigarrow$ WLOG $\mathcal{F}$ const sheaf

Note that:

$\mathcal{F}_{X \setminus U}$, $\mathcal{F}_U$ constructible $\Rightarrow$ $\mathcal{F}$ const.

WLOG $\mathcal{F}$ constant sheaf on U

(Note no WLOG on $\mathcal{G}$, which may have different stratification to $\mathcal{F}$)

$\mathcal{F}_U = M_U$ finite R-module

$$M_U \otimes \mathcal{G}_U \;\&\; RHom(M_U, \mathcal{G}_U)$$
$$\cong \qquad\qquad\qquad \cong$$
$$(\mathcal{F} \otimes \mathcal{G})_U \qquad RHom(M, \mathcal{G})_U$$

lie in $D^b_{Const}(U, K)$

$\square$

Def $\mathcal{F}$ const wrt $\mathcal{I}$ in $Sh(X)$

modified dimension of support

$$\mathrm{mdsupp}(\mathcal{F}) = \sup_{i \in I} \{ \dim(X_i) - \mathrm{grade}(\mathcal{F}_{X_i}) \}$$

Needed for perverse t-structures

Fact

$$-\mathrm{gldim}(R) \leq \mathrm{mdsupp}(\mathcal{F}) \leq \dim(X)$$

(global dimension, sup of proj. dim. of R-modules)

Thm (mdsupp)

(1) mdsupp independent of choice of stratification

(2) dimsupp & mdsupp takes its max in extensions

Good Stratifications

Def $X = \bigsqcup_{i \in I} X_i$ strat $\mathcal{I}$ is good if $\forall \mathcal{F}_i \in D^b_{\mathrm{locsys}}(X)$

$i_i : X_i \hookrightarrow X$, $i_{i*}\mathcal{F}_i \in D^b_{\mathcal{I}}(X)$

Constr.

Ex (1) G-Var X w/ finitely many orb., all conn.

G-Orbits as stratification is good.

(2) X smooth $Z \subseteq X$ div. (irred. comp. $Z_1 \cdots Z_k$)

w/ normal crossings induces a good strat

Why good?

Lem $X = \bigsqcup_{i\in I} X_i$ Strat $\mathcal{X}$ good.

$Y \subseteq X$ locally closed, $Y = \bigsqcup_{i \in I'} X_i$ $I' \subseteq I$

$\iota_Y : Y \hookrightarrow X$:

$\iota_*, \iota_!, \iota^*, \iota^!$ send const. to const.

Determining whether a strat is good is hard! (Whitney regularity conditions)

(Perverse) t-structures

Abstract t-structures $\mathcal{T}$ triangulated

$(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ pair of full subcat

$\mathcal{T}^{\leq n} := \mathcal{T}^{\leq 0}[-n]$, $\mathcal{T}^{\geq n} := \mathcal{T}^{\geq 0}[-n]$

Def $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is called t-structure if

(1) $\mathcal{T}^{\leq -1} \subseteq \mathcal{T}^{\leq 0}$, $\mathcal{T}^{\geq -1} \supseteq \mathcal{T}^{0}$

(2) $\mathrm{Hom}(\underset{\mathcal{T}^{\leq 0}}{\underset{\uparrow}{X}}, \underset{\mathcal{T}^{\geq 0}}{\underset{\uparrow}{Y}}) = 0$

Ex (1) $(D^{\leq 0}(\mathcal{A}), D^{\geq 0}(\mathcal{A}))$

(2) perverse t-structures

Fact (1) is true for shifts

$\mathcal{T}^{\leq n}, \mathcal{T}^{\geq n}$ stable under extensions

(2) Inclusion $(\overset{\leq n}{\text{right}}/\overset{\geq n}{\text{left}})$ adj. to truncation

Thm The heart $\tau^0 = \tau^{\leq 0} \cap \tau^{\geq 0}$ is an abelian category

Idea $g : Y \to X$ in τ^0, find dist. triangle: $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$

f mono $\Leftrightarrow Z \in \tau^0$

$\rightsquigarrow f$ is the kernel of g

$\tau^{\leq 0}, \tau^{\geq 0}$ closed under ext

$\Rightarrow f$ in $\mathrm{Mor}(\tau^0)$

E.g. $\mathcal{A} = D^{\leq 0}(\mathcal{A}) \cap D^{\geq 0}(\mathcal{A})$ is abelian

Perverse t-structures

X is $\mathbb{R}$-Variety

Dg $D^b_{constr}(X)^{\leq 0}$

$$= \{\mathcal{F} \mid \forall i:\ \dim \operatorname{supp} H^i(\mathcal{F}) \leq -i\}$$

$D^b_{constr}(X)^{\geq 0}$

$$= \{\mathcal{F} \mid \forall i:\ \dim \operatorname{supp} H^i(D\mathcal{F}) \leq -i\}$$

Verdier dual of $\mathcal{F}$

is the perverse t-structure

Heart is $\mathrm{Perv}(X)$ perverse sheaves (abelian cat.)

Todo: Prove that it is a t-structure (3.1 Achar)

Note $\mathcal{I}$ good strat. then

$$D^b_{\mathcal{I}}(X)^{\leq 0} = D^b_{constr}(X)^{\leq 0} \cap D^b_{\mathcal{I}}(X)$$

(Analogous for ≥ 0)

$$Perv_{\mathcal{I}} = D^b_{\mathcal{I}}(X)^{\leq 0} \cap D^b_{\mathcal{I}}(X)^{\geq 0}$$

Why perverse?

- abelian cat. (as the heart)
- X smooth conn. $/\mathbb{R}$

$\rightsquigarrow Perv_{\mathcal{I}}(X) = D^{ft}_{Locsys}(X)[\dim X]$
(shifted local syst)

- Compatible w/ Verdier duality
- $f: X \to Y$ finite, $f_*: D^b_{constr}(X) \to D^b_{constr}(Y)$ is t-exact

Variations can be made w.r.t. perversity function

$q: \mathcal{I} \to \mathbb{Z}$

If time permits: Base case $R = k = \mathbb{C}$

↳ loc sys ⇒ loc free

Analytic version is Kobayashi–Hitchin

Narasimhan–Seshadri, X sm proj conn. curve$/\mathbb{C}$ genus $g \geq 2$

$$\mathrm{VecBun}^{st}_{r,0} \xleftrightarrow[\text{Diffeo}]{1:1} \mathrm{Irrep}(\pi_1(X,x), U(r))$$

$$E \longmapsto \left(\rho : [\gamma]_x \longmapsto \rho_\gamma : \mathbb{C}^r \to \mathbb{C}^r \right)$$

unitary and parallel transports vectors over X

Upgrade this to Higgs bundles on LHS:

$$H^1(X,\mathbb{C}) = H^{1,0}(X,\mathbb{C}) \oplus H^{0,1}(X,\mathbb{C})$$
$$= H^1(X,\mathcal{O}_X) \oplus H^0(X,\Omega_X)$$

"replace $\mathbb{C}$ with $GL(r,\mathbb{C})$"

$$\mathrm{HiggsBun}^{st}_{r,0} \xleftrightarrow[\text{Diffeo}]{1:1} \mathrm{Irrep}(\pi_1(X), GL(r,\mathbb{C}))$$

↳ Nice HK structure

If R is general CRing, then replace locally free $\mathcal{O}_X$-mod with local systems