16/10 **TALK 2: INTERSECTION HOMOLOGY VIA CHAIN COMPLEXES**

Basic idea Recovering classical properties ($\nearrow$ Poincaré duality) of homology for singular spaces

$\Rightarrow$ only consider cycles that meet singularities with a controlled defect of transversality

$\hookrightarrow$ allows for local definition which leads to sheaf theory

Goal: interpolate BM homology and homology of transversal chains

§ 1 STRATIFIED SPACES

Def^{n}. If L is a compact Hausdorff space, then the open cone on L is defined as

$$c(L) = L \times [0,1) / L \times \{0\}.$$

A topologically stratified space is defined by induction:

- 0-dimensional stratified space = countable set of points

$\Rightarrow$ n-dimensional stratified space is a Hausdorff topological space X with a filtration

$$X = X_n \supseteq X_{n-1} \supseteq \dots \supseteq X_0 \supseteq X_{-1} = \emptyset$$

by closed subspaces such that for any point $x \in X_i \setminus X_{i-1}$ $\exists\, U_x$ and compact top. stratified space L of dimension $n-i-1$ with a homeomorphism $\varphi : U_x \xrightarrow{\sim} \mathbb{R}^i \times c(L)$ for which

$$\left.\begin{aligned} &\bullet\ \varphi(U_x \cap X_{i+j+1}) \cong \mathbb{R}^i \times c(L_j) \quad \text{for} \quad 0 \leq j \leq n-i-1, \\ &\bullet\ U_x \cap X_j \cong \mathbb{R}^i \times \{x\} \end{aligned}\right\} \text{local normal triviality}$$

d

Each $X_i \setminus X_{i-1}$ is an i-dimensional manifold called the i-dimensional strata of X

Def^{n}. A topological pseudomanifold of dimension n is a top. stratified space X with a filtration such that $X_{n-1} = X_{n-2}$ and so that $X \setminus X_{n-2}$ is dense in X.

Exmp. ① Every manifold is a topological pseudomanifold.

② $S^2 \wedge S^2$ is a top. pseudomanifold.

③ Complex algebraic varieties are top. pseudomanifolds (and PL pseudomanifolds!)

$\hookrightarrow$ Whitney condition $\Rightarrow$ stratification

§ 2 NOTE ON BOREL-MOORE HOMOLOGY

$\hookrightarrow$ relevant for Poincaré duality on non-compact spaces

$$H^i_c(X,\mathbb{Z}) \cong H_{n-i}(X,\mathbb{Z})$$

$$\Big\updownarrow \qquad (X \text{ is an oriented } n\text{-fold})$$

$$H^i(X,\mathbb{Z}) \cong H^{BM}_{n-i}(X,\mathbb{Z})$$

Def^{n}. A piecewise linear (PL) space is a top. space with a class of triangulations $\mathcal{T}$ of locally finite triangulations such that for all triangulations $T \in \mathcal{T}$, any subdivision of T is in $\mathcal{T}$, and any two triangulations $T, T' \in \mathcal{T}$ have a common refinement in $\mathcal{T}$.

Let $(X, \mathcal{T})$ be a PL space.

$$H_i(X, \mathbb{Z}) := H_i(C_\bullet(X))$$

↳ complex of PL chains

$$C_k(X) := \varinjlim_{T \in \mathcal{T}} C_k^T(X)$$

↳ $\sum_{i=1}^{m} a_i \sigma_i$, σ_i are k-subsimplices, $a_i \in \mathbb{Z}$

Similarly, we have $C_k^{BM}(X) := \varinjlim_{T \in \mathcal{T}} C_k^{T,BM}$ of locally finite chains

↳ $\sum_\sigma a_\sigma \sigma$ so that $\forall x \in X \ \exists U_x : \{a_\sigma : a_\sigma \neq 0, U_x \cap \sigma \neq \emptyset\}$ is finite

Then, $H_k^{BM}(X, \mathbb{Z}) = H_k(C_\bullet^{BM}(X))$

Def^{n}. The support of a chain $\xi \in C_k^{BM}(X)$ is $|\xi| = \bigcup_{a_\sigma \neq 0} \sigma$ (it is a closed subset of X)

Let X be a PL pseudomanifold with $\dim X = n$

Def^{n}. A PL i-chain ξ is transverse to the stratification of X if

$$\dim(|\xi| \cap X_{n-k}) = i + (n-k) - n = i - k \quad \text{for all } k \geq 2.$$

Th^{m} [McCrory]. Let $C_\bullet^{tr}(X)$ be locally finite transverse PL chains. Then

$$H_i(C_\bullet^{tr}(X)) \cong H^{n-i}(X, \mathbb{Z})$$

↳ if every chain in X can be moved to make it homologous to a chain transverse to the stratification, we'd have Poincaré duality

! not possible for singular spaces

In general,

$$\begin{tikzcd} H^{n-i}(X, \mathbb{Z}) \arrow[r, "\frown [X]"] \arrow[d, "\wr"] & H_i^{BM}(X, \mathbb{Z}) \arrow[d, "\wr"] \\ H_i(C_\bullet^{tr}(X)) & H_i(C_\bullet^{BM}(X)) \end{tikzcd}$$

→ define intersection homology groups interpolating between $H_i(C_\bullet^{tr}(X))$ and $H_i(C_\bullet^{BM}(X))$

§3 INTERSECTION HOMOLOGY

Def^{n}. A perversity function $\bar{p} : \mathbb{N}_{\geq 2} \to \mathbb{N}$ such that

- $\bar{p}(2) = 0$,
- $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1$ for $k \geq 2$.

Exmp. ① zero perversity $\bar{0}(k) = 0$ for all $k \geq 2$

② top perversity $\bar{t}(k) = k - 2$ for all $k \geq 2$

③ middle perversity $\bar{m}(k) = \left[\frac{k-2}{2}\right]$ for all $k \geq 2$

Remark $\bar{p} = \bar{0}$: $H^{n-i}(X, \mathbb{Z})$
$\wr$
$\bar{p} = \bar{t}$: $H_i^{BM}(X, \mathbb{Z})$

A pair of perversities $\bar{p}, \bar{q}$ are complementary if $\bar{p} + \bar{q} = \bar{t}$

Def^{n}. A PL i-chain $\xi \in C_i^{PL}(X)$ is called $\bar{p}$-allowable if for $k \geq 2$:

① $\dim(|\xi| \cap X_{n-k}) \leq i - k + \bar{p}(k)$,

② $\dim(|\partial \xi| \cap X_{n-k}) \leq i - k - 1 + \bar{p}(k)$.

Let us set $IC_\bullet^{\bar{p}}(X) :=$ $\bar{p}$-allowable finite chains in $C_\bullet^{PL}(X)$

Def^{n}. The i-th intersection homology group with perversity $\bar{p}$ of X is

$$IH_i^{\bar{p}}(X) := H_i(IC_\bullet^{\bar{p}}(X)).$$

Th^{m} [Goresky-MacPherson]. Intersection homology is a topological invariant, i.e. does not depend on the choice of stratification.

Th^{m} [GM] We have $IH_k^{\bar{t}}(X) \cong H_k(X)$

$IH_k^{\bar{0}}(X) \cong H^{n-k}(X)$ $\Rightarrow$ Poincaré duality for complementary perversities

§ 4 EXAMPLE

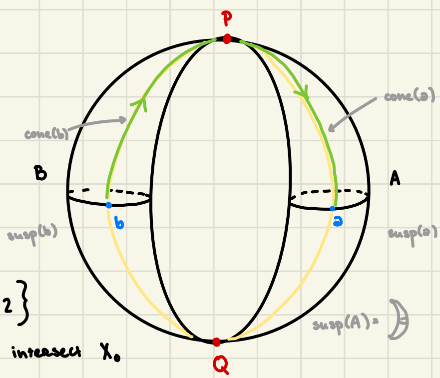
Let $X = \Sigma(S^1 \sqcup S^1)$. The homology groups are:

① $H_0(X, \mathbb{Z}) = \mathbb{Z}$

② $H_1(X, \mathbb{Z}) = \mathbb{Z} = \langle [susp(a) - susp(b)] \rangle$

③ $H_2(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} = \langle [susp(A), susp(B)] \rangle$

↳ stratification $X = X_2 \supseteq X_0 = \{P, Q\}$, perversity $\bar{m}$



- $IC_0^{\bar{m}}(X) = \{ \xi^0 \in C_0(X) : \dim(|\xi^0| \cap X_0) \leq 0 - 2 + \bar{m}(2) = -2 \}$

  $\Rightarrow$ $\bar{m}$-allowable 0-chains cannot intersect X_0

- $IC_1^{\bar{m}}(X) = \{ \xi^1 \in C_1(X) : \dim(|\xi^1| \cap X_0) \leq 1 - 2 + \bar{m}(2) = -1 \}$

- $IC_2^{\bar{m}}(X) = \left\{ \xi^2 \in C_2(X) : \begin{array}{l} \dim(|\xi^2| \cap X_0) \leq 2 - 2 + \bar{m}(2) = 0 \\ \dim(|\partial \xi^2| \cap X_0) \leq 2 - 2 - 1 + \bar{m}(2) = -1 \end{array} \right\}$

  $\Rightarrow$ 2-chains are allowed to intersect P, Q, but their boundaries are not

Hence,

- $IH_0^{\bar{m}}(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ (since is not allowable)
- $IH_1^{\bar{m}}(X, \mathbb{Z}) = 0$
- $IH_2^{\bar{m}}(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} = \langle [susp(A)], [susp(B)] \rangle$