9/10/25

PERVERSE SHEAVES, INTERSECTION COHOMOLOGY & DECOMPOSITION THEOREM

THURSDAYS 10-12

§ 1 OVERVIEW

Slogan „Perverse sheaves are the singular version of local systems."

↳ perverse sheaves live on spaces with singularities:

- algebraic varieties
- PL spaces
- pseudomanifolds

Riemann-Hilbert correspondence (regular holonomic $\mathcal{D}$-modules)

Intersection (co)homology by Goresky - MacPherson

If a complex alg. variety X is singular, some theorems fail: Poincaré duality, Lefschetz theorems, Hodge decomposition, ...

⇒ intersection cohomology groups $IH^\cdot(X)$ ⇝ involves stratifying X

! NOT a homotopy invariant

- in addition, satisfy Mayer-Vietoris and Künneth formula (; if X is projective, we have a pure Hodge structure)
- for nonsingular X: $H^\cdot(X) \xrightarrow{\sim} IH^\cdot(X)$

$\exists$ a geometric construction using cochain complexes (Talk 2?)

But $IH^\cdot(X)$ can also be obtained as the cohomology of $IC_X \in \mathcal{D}^b(X)$

⇒ most important example of a perverse sheaf

intersection cohomology complex

(bounded derived) constructible sheaves on X

Exmp. • X nonsingular $\Rightarrow$ $\mathbb{Q}_X[\dim X]$ is perverse

• $Y \subseteq X$ closed subvariety $\Rightarrow$ $\mathbb{Q}_Y[\dim Y]$ is perverse

• IC_X is perverse

$\Bigg\}$ $P \in D^b(X)$ is perverse if $\dim(\operatorname{supp}(\mathcal{H}^{-i}(P))) \leq i$ and $\dim(\operatorname{supp}(\mathcal{H}^{-i}(\mathbb{D}P))) \leq i$

$\rightsquigarrow$ category $\operatorname{Perv}(X)$: abelian, stable under Verdier duality, Noetherian, Artinian

↳ sidenote: not actual sheaves, but they „glue" like sheaves $\Rightarrow$ form a stack just like ordinary sheaves

Thm [BBDG]. Let $f: X \longrightarrow Y$ be a proper map of varieties, with X nonsingular. Then,

$$f_* \mathbb{Q}_X[\dim X] \simeq \bigoplus_{i \in I} IC_{Y_i}[n_i] \qquad (Y_i \text{ locally closed subvarieties of } Y).$$

Pf. 1) BBDG: via finite fields

2) Saito: mixed Hodge modules

3) de Cataldo-Migliorini: classical Hodge theory

Calculations can be made simpler using a few observations:

• f small, X smooth $\Rightarrow$ $f_* \mathbb{Q}_X[\dim X]$ is an IC

• f semismall, X smooth $\Rightarrow$ $f_* \mathbb{Q}_X[\dim X]$ is a semisimple perverse sheaf (i.e. $n_i = 0$ $\forall i \in I$)

§ 2 AN APPLICATION OF THE DECOMPOSITION THEOREM

Toric geometry $P \subseteq \mathbb{R}^d$ polytope $\rightsquigarrow$ toric variety X_P

↳ $\mathbb{Q}$-smooth if it only has rational singularities

Prop. X_P is $\mathbb{Q}$-smooth $\Leftrightarrow$ P is simplicial.

Let $f_i :=$ # i-dimensional faces of P

$\Rightarrow$ we define the h-polynomial: $h(P,t) = (t-1)^d + f_{d-1}(t-1)^{d-1} + \dots + f_0 = \sum_{k=0}^{d} h_k t^k$

Prop. If P is simplicial, then $h_k = \dim(H^{2k}(X_P, \mathbb{Q}))$.

Moreover, we have Poincaré duality and hard Lefschetz th^{m}:

$$h_k = h_{d-k} \quad \text{for} \quad 0 \le k \le d, \qquad h_{k-1}(P) \le h_k(P) \quad 0 \le k \le d/2 .$$

Exmp. Let C_2 be the square which is simplicial.

$f_0 = 4,\ f_1 = 4$

$\Rightarrow h(C_2,t) = (t-1)^2 + 4(t-1) + 4 = t^2 + 2t + 1$

Hence, $H^0(X_{C_2}, \mathbb{Q}) = \mathbb{Q} = H^4(X_{C_2}, \mathbb{Q})$ and $H^2(X_{C_2}, \mathbb{Q}) = \mathbb{Q}^2$, agrees with other cohomology calculations for $X_{C_2} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Q: Is similar calculation possible for P not simplicial?

$\rightsquigarrow$ redefine

$$h(P,t) = \sum_{F \subset P} g(F,t)(t-1)^{d-1-\dim(F)}, \quad \text{where} \quad g(F,t) = h_0 + (h_1 - h_0)t + \dots + (h_{[d/2]} - h_{[d/2]-1})t^{[d/2]}$$

$$g(\emptyset, t) = h(\emptyset, t) = 1, \quad \dim(\emptyset) = -1$$

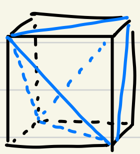
Th^{m}. Let P be a rational polytope. Then,

$$h(P,t) = \sum_{k=0}^{d} \dim(IH^{2k}(X_P, \mathbb{Q}))\, t^k .$$

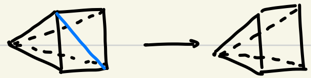
Exmp. Let C_3 be a cube.

$\rightarrow$ subdivision $\tilde{C}_3$ gives a resolution

$f : X_{\tilde{C}_3} \longrightarrow X_{C_3}$



$\tilde{C}_3$ (simplicial)

• local picture : 

C_3 has 6 singular points p_i $(i=1,\dots,6)$; $f^{-1}(p_i) \cong \mathbb{P}^1$

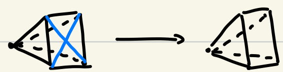
$\Rightarrow h(\tilde{C}_3, t) = (t-1)^3 + 12(t-1)^2 + 18(t-1) + 8 = t^3 + 5t^2 + 5t + 1 = h(C_3, t)$

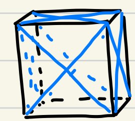
↳ suggests that $H^k(X_{\tilde{C}_3}, \mathbb{Q}) = IH^{k+3}(X_{C_3}, \mathbb{Q})$??

By decomposition th^{m} :

$$\begin{array}{c} f_* \mathbb{Q}_{X_{\tilde{C}_3}}[3] \cong IC_{X_{C^3}} \\ \Big\downarrow {\scriptstyle R\Gamma(-)} \\ H^*(X_{\tilde{C}_3}, \mathbb{Q})[3] \cong IH^*(X_{C^3}, \mathbb{Q}) \end{array}$$

What about a different subdivision?

• local picture: 



C_3' (simplicial)

• subdivision C_3' gives a resolution $g: X_{C_3'} \to X_{C_3}$ (! g is not semismall)

• we have $h(C_3', t) = t^3 + 11t^2 + 11t + 1$; $g^{-1}(p_i) = \mathbb{P}_1 \times \mathbb{P}_1$

By decomposition th^{m} :

$$g_* \mathbb{Q}_{X_{C_3'}}[3] \cong IC_{X_{C_3}} \oplus \underbrace{\Big(\bigoplus_{p_i} IC_{p_i}[n_i] \Big)}_{\text{things supported on points } p_i}$$

$$\cong IC_{X_{C_3}} \oplus \bigoplus_{p_i} \mathbb{Q}_{p_i}[-1] \oplus \bigoplus_{p_i} \mathbb{Q}_{p_i}[1] \qquad \left(\text{note } {}^{p}H^{\pm 1}(g_* \mathbb{Q}_{X_{C_3'}}[3]) = \bigoplus_i \mathbb{Q}_{p_i}\right)$$

$\Rightarrow$ we get: $\dim(H^k(X_{C_3'}, \mathbb{Q})) = \dim(IH^k(X_{C_3}, \mathbb{Q})) + 6$ for $k = 2, 4$, $H^k(X_{C_3'}, \mathbb{Q}) \cong IH^k(X_{C_3}, \mathbb{Q})$ otherwise

$\Rightarrow h(C_3, t) = \sum_i \dim(IH^{2k}(X_{C_3}, \mathbb{Q})) t^k = \sum_i \dim(H^k(X_{C_3'}, \mathbb{Q})) t^k - 6t - 6t^2 = h(C_3', t) - 6t - 6t^2$