EnumGeo 2: Background Material

- Moduli theory (coh sheaves, Grassmannians, closed subschemes, stable curves)
  - Idea behind stacks, Coarse & fine moduli spaces
  - Local models for stacks algebraic, DM, critical loci of $Gr(k, \mathbb{C}^n)$

If time permits, we will look at GIT

- Deformation theory (Alper Appendix C, Ricolfi Appendix A) of coh sheaves, closed subschemes
  - First order deformations
  - Obstruction theories / Higher-order deformations

Moduli theory

Qn: X = sm. proj sch$/\mathbb{C}$, how do we classify on X:

- Coherent sheaves
- Vector bundles
- G-bundles
- Closed 0-dim subschemes

$X = \mathbb{C}^r$, then

- k-dim subspaces

They come parametrized in "families"

e.g. $\begin{array}{c} \mathcal{F} \\ \downarrow \\ X \times T \end{array}$ $\forall t \in T$ pt. $\mathcal{F}_t$ coh. sheaf on X

$\mathcal{C} = Sch/\mathbb{C}$

Families are collections of morph in $\mathcal{C}$ compatible w/ pullbacks $\begin{pmatrix} \mathcal{T} \\ \downarrow \\ T \end{pmatrix}$

Def $\mathcal{M} \in ob(\mathcal{C})$ is a fine moduli space if $\exists$ "universal family" $\begin{matrix} M \\ \downarrow \\ \mathcal{M} \end{matrix}$ s.t.

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d] \\ \mathcal{S} \arrow[r, "\exists!"] & \mathcal{M} \end{tikzcd}$$

pullback in $Sch/\mathbb{C}$

Problem Fine moduli space rarely exists

- Rigidify moduli problem (Stability in GIT)
- Coarse moduli space (delete arrows)
- Too many automorphisms, incorporate them into $Sch/\mathbb{C} \rightsquigarrow$ Stacks

Categories fibered in groupoids

$\mathcal{M} : \mathcal{C}^{op} \longrightarrow$ Groupoid (2-category)

$\begin{pmatrix} S \\ f \downarrow \\ T \end{pmatrix} \rightsquigarrow$ $\mathcal{M}(T)$ = families over T
$\downarrow$ pullback by f
$\mathcal{M}(S)$

Def We call $\mathcal{M}$ a prestack

Def If $(\mathcal{C}, \tau)$ is a Groth. topology (étale, fppf, fpqc), a prestack $\mathcal{M}$ is a stack if compatible w/ gluing & separating.

$\rightsquigarrow$ Can do geometry on stacks!

$Sch/\mathbb{C} \subseteq Stacks/\mathbb{C}$

If $\mathcal{M} \cong \mathrm{Hom}(-, M)$, $M \in \mathrm{ob}(\mathcal{C})$
represents $\mathcal{M}$ as a fine moduli space

E.g. Hilbert schemes (of pts.)
Grassmannians, Quot schemes

Def Stack $\mathcal{M}$ on $(\mathcal{C}, \text{ét})$ is
<u>algebraic</u> if $\exists\, U \longrightarrow \mathcal{M}$, $U \in \mathcal{C}$
is "representable", surjective, smooth

<u>Deligne-Mumford</u> if étale, not just smooth

Algebraic $\Rightarrow$ $\mathcal{M}$ locally at x looks like
$[\mathrm{Spec}\, A / G_x]$
G_x red. lin alg. gp

DM $\Rightarrow$ $[\mathrm{Spec}\, A / G_x]$, G_x finite

<u>Critical loci</u> Goal: Study
$\mathrm{Gr}(k, \mathbb{C}^n)$ locally
(following Ricolfi 3.4)

Def $U \in \mathrm{Sch}/\mathbb{C}$ smooth,
$f : U \longrightarrow \mathbb{C}$ regular,
$\rightsquigarrow df \in H^0(U, T^*U)$
$\cong \mathrm{Hom}_U(\mathcal{O}_U, \Omega_U)$
$Z(df) \subseteq U$ is <u>critical locus</u>
mod. space of interest e.g. $\mathrm{Hilb}^n \mathbb{A}^3$

E.g. $\mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[t])$

$$\mathrm{Spec}(\mathbb{C}[t])/t^n = \mathrm{Crit}(t^{n+1}) = Z((n+1)t^n)$$

E.g. $Gr(k,\mathbb{C}^r)$

$A = H^*(Gr(k,\mathbb{C}^r),\mathbb{C})$ ring

(last week we did $H^*(Gr(2,\mathbb{C}^4),\mathbb{Z})$)

$$A = \mathbb{C}[x_1,\dots,x_r]/I$$

$S \longrightarrow Gr(k,\mathbb{C}^r)$ tautological k-dim bundle

$Q \longrightarrow Gr(k,\mathbb{C}^r)$ quotient $n-k$ bundle

$x_i = c_i(S^*) \in H^{2i}(Gr(k,\mathbb{C}^r),\mathbb{C})$

I generated by $c_t(S^*)\,c_t(Q^*)$

Claim Spec(A) is a critical locus of $\mathbb{A}^r$

- Define

$$-\log(c_t(S^*)) = \sum_{i\geq 0} W_i(x_1\dots x_k)t^i$$

$$W = (-1)^{n+1}W_{n+1} \in \mathbb{C}[x_1\dots x_r]$$

$$Z(dW) = I \rightsquigarrow \text{Claim}$$

Useful for studying def.

1-st order def. of Coh. corresp. to H^1

Result Critical loci $Z(df)$ have a "symmetric obstruction theory" identified with $\mathrm{Hess}(f)$ (Ex. 10.1.10 Ricolfi)

Virtual class $[Z(df)]^{vir}$ is nice!

Deformation theory

$\mathcal{M} : \mathcal{C} \longrightarrow$ Groupoid moduli problem

of
1. Closed subschemes
2. Coherent sheaves / Vec. bun
3. Schemes

How does one object $E_0 \in \mathcal{M}(\mathbb{C})$ deform locally? Or what is $T_E \mathcal{M}$?

$\{ \varsigma \leftarrow \varsigma \rightarrow \varsigma \}$, $\leftarrow \alpha \rightarrow$?

First-order deformation: (or square-zero extension)

$$\mathbb{C}[\epsilon] = \mathbb{C}[t]/t^2$$

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{C}) \arrow[r, hook] \arrow[dr, "E_0"'] & \mathrm{Spec}(\mathbb{C}[t]/t^2) \arrow[d, "E"] \\ & \mathcal{M} \end{tikzcd}$$

1. $X \in Sch/\mathbb{C}$, $Z_0 \subseteq X$ closed subscheme

Def $Z \subseteq X \times_{\mathbb{C}} \mathrm{Spec}(\mathbb{C}[\epsilon]) = X_{\mathbb{C}[\epsilon]}$ is 1-st order def, if Z flat$/\mathbb{C}[\epsilon]$ & $Z_0 = Z \times_{\mathbb{C}[\epsilon]} \mathbb{C}$

② E_0 coh sheaf on X

Def

$$\begin{tikzcd} E_0 \arrow[r, hook, "\alpha"] \arrow[d, dash] & E \arrow[d, dash] \\ X \arrow[r, hook] & X_{\mathbb{C}[\epsilon]} \end{tikzcd}$$

E coh sheaf on $X_{\mathbb{C}[\epsilon]}$ is 1-st order deformation

More generally: A = Artinian ring$/\mathbb{C}$ (important!)

$\rightsquigarrow$ Deformations over A

Lemma ① $Z_0 \subseteq X$ defined by ideal sheaf $I_0 \subseteq \mathcal{O}_X$, then

$$\{\text{first order def } Z_0\}/_{\text{isom}} \simeq H^0(Z_0, N_{Z_0/X})$$

$$\mathrm{Hom}(I_0/I_0^2, \mathcal{O}_{Z_0})$$

Normal sheaf.

GO TO END
DO EXAMPLE

Lemma ② $E_0 \to X$

$$\{\text{first order def. } (E,\alpha)\}/_{\text{isom}} \simeq \mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0)$$

$$\mathrm{S}\!/\!/$$

$$H^1(X, \mathcal{E}nd_{\mathcal{O}_X}(E_0))$$

may ignore

E.g. ③ Smooth!

$T_C\mathcal{M}_g = H^1(C, T_C)$
$= H^0(C, \Omega^{\otimes 2}_{C/\mathbb{C}})$

Obstruction theory

$\mathcal{M}$ moduli problem (e.g. ① ② ③)

$A' \twoheadrightarrow A$ artinian over $\mathbb{C}$

(e.g. $\mathbb{C}[t]/t^3 \twoheadrightarrow \mathbb{C}[t]/t^2$)

Q: $E \in \mathcal{M}(A)$, is there a deformation E' of E over A'? How to classify them?

A: Vanishing of some obstruction class

If A, A' are local s.t. $A/m_A \cong A'/m_{A'} \cong \mathbb{C}$

$J = \ker(A' \twoheadrightarrow A)$ s.t. $m_{A'} J = 0$

$\left(\Rightarrow \text{square-zero extension } J^2 = 0\right)$

E is deformation of $E_0 := E|_{\mathbb{C}}$ over A

(Perfect obstruction theory)

$$\begin{tikzcd} \operatorname{Spec}(A) \arrow[r] \arrow[d] & \operatorname{Spec}(\mathbb{C}) \arrow[d, "E_0"] \\ \operatorname{Spec}(A') \arrow[r] \arrow[ur, dashed] & \mathcal{M} \end{tikzcd}$$

0 obstruction,

$\Downarrow$

Lifting crit for smoothness $\Rightarrow$ $\mathcal{M}$ smooth at E_0

Proposition 1

X scheme $/\mathbb{C}$ w/ separated $\Delta : X \to X \times X$

This stuff & $Z \subseteq X_A$ closed & flat over A, $Z_0 = Z \times_A \mathbb{C}$

(1) If $\exists$ deformations over A' then they form torsor w.r.t. $\operatorname{Ext}^0_{\mathcal{O}_X}(I_{Z_0}, \mathcal{O}_{Z_0} \otimes_{\mathbb{C}} J)$

(2) $\exists\, ob_Z \in \operatorname{Ext}^1_{\mathcal{O}_X}(I_{Z_0}, \mathcal{O}_{Z_0} \otimes_{\mathbb{C}} J)$ s.t. $\exists$ def over A' $\Longleftrightarrow$ $ob_Z = 0$

Proposition ② X scheme $/\mathbb{C}$

Same as before w/ A, A', J,

E coh sheaf on X_A

(1) If $\exists$ def. E' over A', the group of aut. of E' is bij. to $\mathrm{Ext}^0_{\mathcal{O}_X}(E_0, E_0 \otimes J)$

(2) If $\exists$ deformations E' over A', isom. classes then they form torsor w.r.t. $\mathrm{Ext}^1_{\mathcal{O}_X}(E_0, E_0 \otimes_{\mathbb{C}} J)$

(3) $\exists$ $ob_E \in \mathrm{Ext}^2_{\mathcal{O}_X}(E_0, E_0 \otimes_{\mathbb{C}} J)$ s.t. $\exists$ def over A' $\Longleftrightarrow$ $ob_E = 0$

APPLICATION of LEMMA ①

$M = \mathrm{Hilb}^4 \mathbb{A}^3$ irred of dim 12

(Hilb scheme of 3 points in $\mathbb{A}^4$)

$I_X = (x, y, z)^2$, $Z \subseteq M$

induces $p : \mathcal{O}_{\mathbb{A}^3} \twoheadrightarrow \mathcal{O}_Z$

$$\dim(T_p M) \overset{\text{Lemma ①}}{=} \dim(H^0(Z, N_{Z/\mathbb{A}^3})) = 18 > 12$$

M is singular at Z

& $ob_Z \neq 0$