4. Resolution of $\mathbb{Z}$ over $\mathbb{Z}G$

G group, we can always think of $\mathbb{Z}$ as a G-module by taking the trivial action $g \cdot a = a \ \forall g \in G$.

We will see that free resolutions of $\mathbb{Z}$ arise from free actions of G on <u>contractible</u> complexes.

DEF: a G-COMPLEX X is a CW-complex together with $G \curvearrowright X$ permuting cells.

In particular $\forall g \in G$ we have a homeo. $x \longmapsto g \cdot x$ s.t. $\forall \sigma$ cell of X then $g\sigma$ is a cell.

Recall: given a CW-complex X we have the associated cellular chain complex

$$C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

where $C_k \cong \mathbb{Z}^{a_k}$, with $a_k = \#\{k\text{-cells in } X\}$.

Note that, if X is a G-complex then the action $G \curvearrowright X$ induces an action $G \curvearrowright C_*(X)$, hence $C_k(X)$ is a G-mod. for all k and the augmentation map $\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$ is a map of G-mod.

$\hookrightarrow$ $\varepsilon(v) = 1 \ \forall v$ 0-cell of X.

DEF: a G-complex X is FREE if the action of G on the cells is free, i.e. $g \cdot \sigma \neq \sigma \ \forall g \neq 1 \in G$.

$\hookrightarrow$ the chain modules $C_k(X)$ become free G-modules:

the group G acts freely on the $\mathbb{Z}$-basis of $C_k(X)$, hence we get a free $\mathbb{Z}G$-module with one basis element for each orbit.

PROP: let X be a contractible free G-complex. Then the augmented cellular chain complex of X is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$

<u>Proof:</u> if X is contractible we have $H_m(X) = H_m(\{pt.\}) = \begin{cases} 0 & m \geq 1 \\ \mathbb{Z} & m = 0 \end{cases}$

In other words, the augmented chain complex is exact:

$$\cdots \longrightarrow C_m(X) \xrightarrow{\partial_m} C_{m-1}(X) \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

but each $C_k(X)$ is a free $\mathbb{Z}G$-mod. $\Rightarrow$ we have a free resolution. $\square$

Covering spaces

$$\begin{tikzcd} \tilde{X} \arrow[rr, "g"] \arrow[dr, "p"'] & G & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

<u>Review:</u> given a covering $p : \tilde{X} \longrightarrow X$, a DECK TRANSFORMATION of p is a homeo. $g : \tilde{X} \longrightarrow \tilde{X}$ s.t.

Let G be the group of deck trasformations, then $G \curvearrowright \tilde{X}$ freely.

The covering p is REGULAR if it satisfies the following equivalent conditions:

- G acts transitively on $p^{-1}(x) \ \forall x \in X$.
- the image of $\pi_1 \tilde{X} \longrightarrow \pi_1 X$ is normal in $\pi_1 X$.

- $\forall\, \omega$ closed loop in X, if one lift is closed then all are.

PROP: given a regular cover p, we have:

$$G \cong \pi_1 X \big/ \pi_1 \tilde{X}$$

Note: a regular G-cover of X (or principal G-bundle over X) is a covering $p: \tilde{X} \to X$ s.t. $G \curvearrowright \tilde{X}$ freely and transitively on the fibers.

PROP: given a regular covering $p: \tilde{Y} \to Y$, if Y is a CW-complex then $\tilde{Y}$ inherits a CW-structure s.t. the action of the grop of deck transformations permutese the cells.

$\hookrightarrow$ G acts freely + transitively on $p^{-1}(x)$ $\Rightarrow$ $\tilde{Y}$ is a free G-complex and $C_*(\tilde{Y})$ has free $\mathbb{Z}G$-mod. with bases cells of Y.

DEF: a CW-complex Y satisfying:

i. Y is connected
ii. $\pi_1 Y \cong G$
iii. the universal cover $X \xrightarrow{p} Y$ has $H_i X = 0$ $\forall i \geq 2$

is called an EILENBERG-MACLANE complex or $K(G,1)$-COMPLEX.

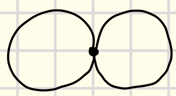
RMK: iii) can be replaced by $\pi_i Y = 0$ $\forall i \geq 2$. Thus we have aspherical complexes.

Note that, if Y is a $K(G,1)$ then the universal cover $p: X \to Y$ is a regular cover with deck transformation group $\pi_1 Y / \{*\} \cong \pi_1 Y \cong G$.

PROP: if Y is a $K(G,1)$ then the augmented chain complex of the universal cover is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

example: $G = F(S)$ free group with generating set S

$Y = \bigvee_{s \in S} S^1_s$ bouquet of circles $\rightsquigarrow$ 1-dim. CW-complex



$\hookrightarrow$ recall that $\pi_1 Y = F(S)$ and its universal cover is given by the Cayley graph of $F(S)$

- take a vertex $x_0 \in X$ $\Rightarrow$ this gives the unique G-orbit of 0-cells $\Rightarrow$ $C_0(X) \cong \mathbb{Z}G$
- $C_1(X)$ has basis given by oriented 1-cells e_s, which can be taken to have vertex x_0

$\hookrightarrow$ we have $\partial e_s = s x_0 - x_0 = (s-1)x_0$, therefore we get

$$0 \longrightarrow \mathbb{Z}G^{|S|} \xrightarrow{(s-1)} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

5. Standard resolution

Let X be a contractible simplicial complex with $G \curvearrowright X$ simplicially. In order to have a free action of G, and consequently a free resolution, we take the ORDERED CHAIN COMPLEX $C'_*(X)$, defined by:

$C'_m(X)$ is the $\mathbb{Z}$-mod. with basis ordered $(m+1)$-tuples $(v_0, \dots, v_m)$ of vertices of X s.t. $\{v_0, \dots, v_m\}$ is a simplex

Take X to be the "simplex" spanned by G, i.e. elements of G give vertices of X and every finite subset of G is a simplex of X. The group G acts by left multiplication.

We obtain a free resolution $F_* := C'_*(X)$ of $\mathbb{Z}$ over $\mathbb{Z}G$ called the STANDARD RESOLUTION:

- F_m is the free $\mathbb{Z}$-mod. generated by $(m+1)$-tuples $(g_0, \dots, g_m)$
- $\partial : F_m \longrightarrow F_{m-1}$ is given by $\partial := \sum_{i=0}^{m} (-1)^i d_i$, with $d_i(g_0 \dots g_m) = (g_0 \dots \hat{g_i} \dots g_m)$.

As $\mathbb{Z}G$-mod. F_m has basis $(m+1)$-tuples of the form $(1, \dots)$. It is useful to use the bar notation

$$[g_1 | g_2 | \dots | g_m] := (1, g_1, g_1 g_2, \dots).$$

Then

$$d_i([g_1 | \dots | g_m]) = \begin{cases} g_1 \cdot [g_2 | \dots | g_m] & , i = 0 \\ [g_1 | \dots | g_i g_{i+1} | \dots | g_m] & , 0 < i < m \\ [g_1 | \dots | g_{m-1}] & , i = m \end{cases}$$

6. Periodic resolutions

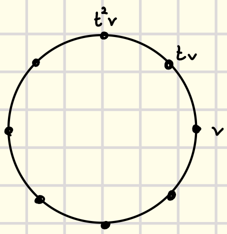
Suppose X is a free G-complex homeomorphic to S^{2k-1}, then G acts trivially on $H_{2k-1}(X) = \mathbb{Z}$. This gives the following exact sequence of G-modules: $0 \to \mathbb{Z} \xrightarrow{\eta} C_{2k-1} \to \dots \to C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$

Splicing together infinite copies, we obtain a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ of period $2k$:

$$\dots \to C_1 \to C_0 \xrightarrow{\eta\varepsilon} C_{2k-1} \to \dots \to C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

example: G = cyclic group of order m

G acts freely on S^1, seen as m 0-cells + m 1-cells

$C(8) \curvearrowright$ 

$\rightsquigarrow \quad 0 \to \mathbb{Z} \xrightarrow{\eta} \underset{C_1}{\mathbb{Z}G} \xrightarrow{t\cdot} \underset{C_0}{\mathbb{Z}G} \xrightarrow{\varepsilon} \mathbb{Z} \to$

7. Uniqueness of resolution

Setting: R ring

M an R-module

$\rightsquigarrow$ Claim: all free resolutions of M are homotopy equivalent

DEF: a R-module P is **PROJECTIVE** if the functor $\mathrm{Hom}_R(P,-) : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$ is exact.

Equivalently, P is projective iff $\forall\, \alpha : A \to B$ surjection the map $\alpha\circ - : \mathrm{Hom}(P,A) \to \mathrm{Hom}(P,B)$ is surjective.

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \quad \text{exact}$$

$$\Big\downarrow \mathrm{Hom}(P,-)$$

$$0 \to \mathrm{Hom}(P,A) \xrightarrow{\alpha\circ -} \mathrm{Hom}(P,B) \xrightarrow{\beta\circ -} \mathrm{Hom}(P,C) \to 0 \quad \text{exact}$$

In particular, we can always complete the diagram

$$\begin{tikzcd} & P \arrow[dl, "\psi"'] \arrow[d, "\varphi"] \arrow[dr, "0"] & \\ M \arrow[r, "i"'] & M' \arrow[r, "j"'] & M'' \end{tikzcd}$$

exact row:

LEMMA: free modules are projective.