Stability Criteria

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References:

Hoskins, Victoria - Moduli Problems and Geometric Invariant Theory

0. Setting

G reductive group

X projective scheme

$G \curvearrowright X$

$\mathcal{L}$ ample linearization

Want: $X^{ss}(L) \subset X$

Need: G-invariant sections of all $\mathcal{L}^{\otimes r}$

Problem: often hard/impossible to compute

Answer: criteria to determine semistability

↳ the topological criterion

↳ the Hibert-Mumford criterion

Recall: simplification

$X \subset \mathbb{P}^n$ and linear G-action $\left(\begin{array}{l} \text{for some } r, \text{ embedding } X \subset \mathbb{P}^n = \mathbb{P}(H^0(X, \mathcal{L}^{\otimes r})^*) \\ \text{s.t. } \mathcal{O}_{\mathbb{P}^n}(1)|_X = \mathcal{L}^{\otimes r} \text{ and } G \text{ acts linearly on } \mathbb{P}^n \end{array} \right)$

G acts via $G \longrightarrow GL_{n+1}$ $\rightsquigarrow$ lifts to $G \curvearrowright \tilde{X} \subset \mathbb{A}^{n+1}$ affine cone

$R(X) = \mathcal{O}(\tilde{X})$

1. Topological Criterion

Proposition:

Let $x \in X(k)$ and choose non-zero lift $\tilde{x} \in \tilde{X}(k)$ of x.

1. x semistable $\Leftrightarrow 0 \notin \overline{G \cdot \tilde{x}}$.

2. x stable $\Leftrightarrow \dim G_{\tilde{x}} = 0$ and $G \cdot \tilde{x}$ closed in $\tilde{X}$.

Proof:

1. $x \in X^{ss}(k) \longrightarrow \exists f \in R(X)^G$ s.t. $f(x) \neq 0 \rightarrow f(\tilde{x}) \neq 0 \Longrightarrow$
$$f(\tilde{x}) = f(\overline{G \cdot \tilde{x}})$$
$\longrightarrow f$ separates the closed subschemes $\overline{G \cdot \tilde{x}}$ and $0 \Longrightarrow$
$\Longrightarrow \overline{G \cdot \tilde{x}} \cap 0 = \emptyset$.

This argument can be reversed to get $\Leftarrow$.

2. [See Hoskins]

$\square$

2. The Hilbert-Mumford Criterion

projective scheme $X \subset \mathbb{P}^n$ with linear G-action of a reductive group G.

Definition

A 1-parameter subgroup (1-PS) of G is a non-trivial group homomorphism $\lambda : \mathbb{G}_m \longrightarrow G$.

Notation:

For $x \in X(k)$ and λ 1-PS, let
$$\lambda_x : \mathbb{G}_m \longrightarrow X, \quad t \longmapsto \lambda(t) \cdot x$$

Construction:

X projective $\Longrightarrow$ proper over Spec$k$ $\xRightarrow[\text{criterion for properness}]{\text{valuation}}$ $\exists!\, \hat{\lambda}_x : \mathbb{P}^1 \longrightarrow X$ s.t.

$$\begin{tikzcd} t \arrow[d, mapsto] & \mathbb{G}_m \arrow[r, "\lambda_x"] \arrow[d] & X \arrow[d] \\ {[1:t]} & \mathbb{P}^1 \arrow[r] \arrow[ur, dotted, "\exists!\, \hat{\lambda}_x"] & \operatorname{Spec} k \end{tikzcd}$$

with $t \in \mathbb{A}^1 - \{0\} = \mathbb{G}_m$.

$$\lim_{t \to 0} \lambda(t) \cdot x := \hat{\lambda}_x([1:0])$$

$\hookrightarrow$ $y := \lim_{t \to 0} \lambda(t) \cdot x$ fixed by $\lambda(\mathbb{G}_m)$-action $\Longrightarrow$ on the fiber over y of line bundle $\mathcal{O}(1) := \mathcal{O}_{\mathbb{P}^n}(1)|_X$ the group $\lambda(\mathbb{G}_m)$ acts by character $t \mapsto t^r$

$$\lim_{t \to \infty} \lambda(t) \cdot x := \hat{\lambda}_x([0:1])$$

$\hookrightarrow$ unnecessary as $\lim_{t \to \infty} \lambda(t) \cdot x = \lim_{t \to 0} \lambda^{-1}(t) \cdot x$

Definition:

Hilbert-Mumford weight of the action of the 1-PS λ on $x \in X(k)$

$$\mu^{\mathcal{O}(1)}(x,\lambda) := r$$

↳ weight of $\lambda(\mathbb{G}_m)$ on the fiber $\mathcal{O}(1)_y$

More explicitely:

tautological bundle $= \mathcal{O}_{\mathbb{P}^1}(-1)$ is the blow up of the affine cone $\mathbb{A}^{n+1}$ over $\mathbb{P}^n$ at the origin

pick a non-zero lift $\tilde{x} \in \tilde{X}$ of $x \in X$

$\lambda_{\tilde{x}} := \lambda(-)\cdot\tilde{x} : \mathbb{G}_m \longrightarrow \tilde{X}$ (might not extend to $\mathbb{P}^1$ as $\tilde{X}$ not proper)

↳ If it extends to 0 (or ∞) call the limits $\lim_{t\to 0} \lambda(t)\cdot\tilde{x}$

$\left(\lim_{t\to\infty} \lambda(t)\cdot\tilde{x}\right)$

↳ any pt. in boundary $\overline{\lambda_{\tilde{x}}(\mathbb{G}_m)} \setminus \lambda_{\tilde{x}}(\mathbb{G}_m)$ must be equal to either of these pts.

$\lambda(\mathbb{G}_m) \curvearrowright \mathbb{A}^{n+1}$ linear $\Rightarrow$ diagonalizable $\Rightarrow$ pick basis $e_0, \dots, e_n \in k^{n+1}$ s.t.

$\lambda(t)\cdot e_i = t^{r_i} e_i$

for $r_i \in \mathbb{Z}$.

$$\tilde{x} = \sum_{i=0}^{n} x_i e_i \longrightarrow \lambda(t)\cdot\tilde{x} = \sum t^{r_i} x_i e_i$$

$\lambda\text{-wt}(x) := \{r_i : x_i \neq 0\}$

↳ independent of choice of lift $\tilde{x}$

Definition:

Hilbert-Mumford weight of x at $\lambda \equiv \mu(x,\lambda) := -\min\{r_i : x_i \neq 0\}$.

Properties:

1. $\mu(x,\lambda)$ = unique integer s.t. $\lim_{t\to 0} t^{\mu}\lambda(t)\cdot\tilde{x}$ exists and is non-zero.

2. $\mu(x,\lambda^n) = n\mu(x,\lambda)$ for positive n.

3. $\mu(g\cdot x, g\lambda g^{-1}) = \mu(x,\lambda)$ $\forall g\in G$.

4. $\mu(x,\lambda) = \mu(y,\lambda)$ where $y = \lim_{t\to 0}\lambda(t)\cdot x$.

Lemma:

$\mu^{\mathcal{O}(1)}(x,\lambda) = \mu(x,\lambda)$.

Proof:

Choosing coords. as above,

$$\lambda(t)\cdot\tilde{x} = \lambda(t)\cdot(x_0,\dots,x_n) = (t^{r_0}x_0,\dots,t^{r_n}x_n).$$

Then
$$\tilde{y} := \lim_{t\to 0} t^{\mu(x,y)}\lambda(t)\cdot\tilde{x} = (y_0,\dots,y_n)$$
with $y_i = \begin{cases} x_i, & \text{if } r_i = -\mu(x,\lambda) \\ 0, & \text{o.w.} \end{cases}$

Thus $\lambda(t)\cdot\tilde{y} = t^{-\mu(x,y)}\tilde{y} \underset{\Leftarrow}{\Longrightarrow}$ weight of the $\lambda(\mathbb{G}_m)$-action on $\tilde{y} = -\mu(x,\lambda) \Rightarrow$

$\tilde{y}$ lies over $y := \lim_{t\to 0}\lambda(t)\cdot x$

$\underset{\Leftarrow}{\Longrightarrow}$ weight of the $\lambda(\mathbb{G}_m)$-action on $\mathcal{O}_{\mathbb{P}^n}(-1)_y = -\mu(x,\lambda) \Longrightarrow$

$\mathcal{O}_{\mathbb{P}^n}(-1)$ is the blow up of $\mathbb{A}^{n+1}$ at 0

$\Longrightarrow \mu^{\mathcal{O}(1)}(x,\lambda)$ = weight of the $\lambda(\mathbb{G}_m)$-action on $\mathcal{O}_{\mathbb{P}^n}(1)_y = \mu(x,y)$ $\square$

Lemma

- $\mu(x,\lambda)<0 \iff \tilde{x}=\sum_{r_i>0} x_i e_i \iff \lim_{t\to 0} \lambda(t)\cdot\tilde{x}=0$

- $\mu(x,\lambda)=0 \iff \tilde{x}=\sum_{r_i\geq 0} x_i e_i$ and $\exists r_i=0$ s.t. $x_i\neq 0$ $\iff \exists \lim_{t\to 0}\lambda(t)\cdot\tilde{x}\neq 0$

- $\mu(x,\lambda)>0 \iff \tilde{x}=\sum_{r_i} x_i e_i$ and $\exists r_i<0$ s.t. $x_i\neq 0$ $\iff \nexists \lim_{t\to 0}\lambda(t)\cdot\tilde{x}$

By applying this to λ^{-1}, we get:

- $\mu(x,\lambda^{-1})<0 \iff \tilde{x}=\sum_{r_i<0} x_i e_i \iff \lim_{t\to\infty}\lambda(t)\cdot\tilde{x}=0$

- $\mu(x,\lambda^{-1})=0 \iff \tilde{x}=\sum_{r_i\leq 0} x_i e_i$ and $\exists r_i=0$ s.t. $x_i\neq 0$ $\iff \exists \lim_{t\to\infty}\lambda(t)\cdot\tilde{x}\neq 0$

- $\mu(x,\lambda^{-1})>0 \iff \tilde{x}=\sum x_i e_i$ and $\exists r_i>0$ s.t. $x_i\neq 0$ $\iff \nexists \lim_{t\to\infty}\lambda(t)\cdot\tilde{x}$

Lemma:

1. x is semistable for the action of $\lambda(\mathbb{G}_m) \iff \mu(x,\lambda)\geq 0$ and $\mu(x,\lambda^{-1})\geq 0$

2. x is stable for the action of $\lambda(\mathbb{G}_m) \iff \mu(x,\lambda)>0$ and $\mu(x,\lambda^{-1})>0$

Proof:

Follows from previous lemma and topological criterion, as the only boundary points of $\lambda(\mathbb{G}_m)\cdot\tilde{x}$ are $\lim_{t\to 0}\lambda(t)\cdot\tilde{x}$ and $\lim_{t\to\infty}\lambda(t)\cdot\tilde{x}$.

- semistable $\iff 0\notin\overline{\lambda(\mathbb{G}_m)\cdot\tilde{x}} \iff$ limits $\nexists$ or $\neq 0 \iff \mu(x,\lambda),\mu(x,\lambda^{-1})\geq 0$

- stable $\iff \underbrace{\text{semistable} + \text{closed orbit}}_{\Updownarrow\ \nexists\text{ limits}\ \Updownarrow\ \mu(x,\lambda),\,\mu(x,\lambda^{-1})>0} + \underbrace{\dim\lambda(\mathbb{G}_m)_{\tilde{x}}=0}_{\text{follows immediately because if the limits }\nexists\text{, then }\tilde{x}\text{ is not fixed by }\lambda(\mathbb{G}_m)}$

$\square$

Remark:

(semi)stable for $G \Rightarrow$ (semi)stable for all subgroups $H \subset G$

$\forall x \in X(k)$: x semistable $\Rightarrow \mu(x, \lambda) \geq 0$ $\forall$ 1-PSs λ of G

x stable $\Rightarrow \mu(x, \lambda) > 0$ $\forall$ 1-PSs λ of G

Theorem: Hilbert-Mumford Criterion

G reductive group acting linearly on proj scheme $X \subset \mathbb{P}^n$, $x \in X(k)$

$$x \in X^{ss} \Longleftrightarrow \mu(x, \lambda) \geq 0 \quad \forall \text{ 1-PSs } \lambda \text{ of } G$$

$$x \in X^{s} \Longleftrightarrow \mu(x, \lambda) > 0 \quad \forall \text{ 1-PSs } \lambda \text{ of } G$$

$\rightsquigarrow$ equivalent by topological criterion and previous lemma

Theorem: Fundamental Theorem in GIT

G reductive group acting on $\mathbb{A}^{n+1}$

If $x \in \mathbb{A}^{n+1}$ is a closed pt and $y \in \overline{G \cdot x}$, then there is a 1-PS λ of G s.t. $\lim_{t \to 0} \lambda(t) \cdot x = y$.

Example:

$G = \mathbb{G}_m \curvearrowright X = \mathbb{P}^n$ by $t \cdot [x_0 : \cdots : x_n] = [t^{-1}x_0 : tx_1 : \cdots : tx_n]$

$\lambda(t) = t$

$$\lambda(t) \cdot \tilde{x} = (t^{-1}x_0, tx_1, \ldots, tx_n)$$

$$\lim_{t \to 0} \lambda(t) \cdot \tilde{x} = \begin{cases} \nexists, & \text{if } x_0 \neq 0 \\ 0, & \text{if } x_0 = 0 \end{cases}$$

$$\mu(x, \lambda) = \begin{cases} -(-1) = 1 > 0, & \text{if } x_0 \neq 0 \\ -1 < 0, & \text{if } x_0 = 0 \end{cases}$$

$\lambda^{-1}(t) = t^{-1}$

$$\lambda^{-1}(t) = (tx_0, t^{-1}x_1, \ldots, t^{-1}x_n)$$

$$\lim_{t \to \infty} \lambda(t) \cdot \tilde{x} = \lim_{t \to 0} \lambda^{-1}(t) \cdot \tilde{x} = \begin{cases} \nexists, & \text{if } x_i \neq 0 \text{ for some } i \geq 1 \\ 0, & \text{if } x_1 = \cdots = x_n = 0 \end{cases}$$

$$\mu(x, \lambda^{-1}) = \begin{cases} -(-1) = 1 > 0, & \text{if } x_i \neq 0 \text{ for some } i \geq 1 \\ -1 < 0, & \text{if } x_1 = \cdots = x_n = 0 \end{cases}$$

$$X^{ss}(k) = X^s(k) = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n \mid x_0 \neq 0 \text{ and } x_i \neq 0 \text{ for some } i \geq 1\}$$

$\hookrightarrow$ coincides with what we got last week:

$R(X) = k[x_0 : \cdots : x_n]$

$R(X)^G = k[x_0x_1 : \cdots : x_0x_n] \cong k[y_0 : \cdots : y_{n-1}]$

$X = \mathbb{P}^n \dashrightarrow \operatorname{Proj} R(X)^G = \mathbb{P}^{n-1}$

$[x_0 : \cdots : x_n] \longmapsto [x_0x_1 : \cdots : x_0x_n]$

$X^s(k) = X^{ss}(k) = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : \exists i \geq 1 \text{ s.t. } x_0x_i \neq 0\} =$

$= \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_0 \neq 0 \text{ and } \exists i \geq 1 \text{ s.t. } x_i \neq 0\}$

$\cong \mathbb{A}^n - \{0\}$

Exercise:

$G = \mathbb{G}_m \curvearrowright X = \mathbb{P}^2$ by $t[x:y:z] = [tx : y : t^{-1}z]$

$\lambda(t) = t$

$$\lambda(t)\cdot\tilde{x} = (tx, y, t^{-1}z)$$

$$\lim_{t\to 0} \lambda(t)\cdot\tilde{x} = \begin{cases} \nexists, & \text{if } z \neq 0 \\ (0,y,0), & \text{if } z = 0 \end{cases}$$

$$\mu(x,\lambda) = \begin{cases} -(-1) = 1 > 0, & \text{if } z \neq 0 \\ -0 = 0, & \text{if } z = 0 \text{ and } y \neq 0 \\ -1 < 0, & \text{if } z = 0 \text{ and } y = 0 \end{cases}$$

$\lambda^{-1}(t) = t^{-1}$

$$\lambda^{-1}(t)\cdot\tilde{x} = (t^{-1}x, y, tz)$$

$$\lim_{t\to \infty} \lambda(t)\cdot\tilde{x} = \lim_{t\to 0} \lambda^{-1}(t)\cdot\tilde{x} = \begin{cases} \nexists, & \text{if } x \neq 0 \\ (0,y,0), & \text{if } x = 0 \end{cases}$$

$$\mu(x,\lambda^{-1}) = \begin{cases} -(-1) = 1 > 0, & \text{if } x \neq 0 \\ -0 = 0, & \text{if } x = 0 \text{ and } y \neq 0 \\ -1 < 0, & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$

$X^{ss}(k) = \{[x:y:z] \in \mathbb{P}^2 \mid (x \neq 0 \text{ and } z \neq 0) \text{ or } (y \neq 0)\}$

$\nparallel \rightsquigarrow X^s(k) = X^{ss}(k) \setminus \{[0:1:0] \in \mathbb{P}^2\} \rightsquigarrow$ makes sense because $[0:1:0]$ is a fixed point (stabilizer = whole group)

$X^s(k) = \{[x:y:z] \in \mathbb{P}^2 \mid x \neq 0 \text{ and } z \neq 0\}$

3. The Hilbert-Mumford Criterion for Ample Linearizations

Definition:

Hilbert-Mumford weight of a 1-PS λ and $x \in X(k)$ wrt L is

$\mu^L(x, \lambda) := r =$ weight of the $\lambda(\mathbb{G}_m)$-action on the fiber L_y over the fixed pt $y := \lim_{t \to 0} \lambda(t) \cdot x$

Theorem: Hilbert-Mumford Criterion for ample linearization

G reductive group acting on projective scheme X

L ample linearization

$x \in X(k)$

$$x \in X^{ss}(L) \iff \mu^L(x, \lambda) \geq 0 \ \forall \text{ 1-PSs } \lambda \text{ of } G$$

$$x \in X^{s}(L) \iff \mu^L(x, \lambda) > 0 \ \forall \text{ 1-PSs } \lambda \text{ of } G$$

Proof:

L ample $\rightarrow L^{\otimes n}$ very ample for some n

enough to check for $L^{\otimes n}$ as $\mu^{L^{\otimes n}}(x, \lambda) = n\mu^L(x, \lambda)$

$L^{\otimes n}$ induces G-equivariant embedding $i: X \hookrightarrow \mathbb{P}^n$ s.t. $L \cong i^* \mathcal{O}_{\mathbb{P}^n}(1)$.

Result follows from linear setting.

$\square$

4. Proof of the Fundamental Theorem in GIT

suffices to prove this weaker version

Theorem:

G reductive group acting linearly on $\mathbb{A}^n$, $z \in \mathbb{A}^n(k)$.

If 0 lies in the $\overline{G \cdot z}$, then $\exists$ 1-PS λ of G s.t. $\lim_{t \to 0} \lambda(t) \cdot z = 0$.

Proof:

Assume $0 \in \overline{G \cdot z}$.

- Step 1:

  lemma $\Rightarrow \exists$ irreducible curve (not necessarily complete or smooth) $C_1 \subset G \cdot z$ s.t. $0 \in \overline{C_1}$.

- Step 2:

  lemma $\Rightarrow \exists$ smooth projective curve C, a rational map $p : C \dashrightarrow G$ and $c_0 \in C(k)$ s.t.

  $$\lim_{c \to c_0} p(c) \cdot z = 0.$$

- Step 3:

  C smooth proper curve $\Rightarrow$ completion of local ring $\mathcal{O}_{C,c_0}$ of C at c_0

  $\cong$ $k[[t]]$ formal power series

  $\downarrow$

  field of fractions $\cong k((t))$ Laurent series (only finitely many terms of deg <0)

$p : C \dashrightarrow G$ induces morphism $q : K \longrightarrow G$
defined in punctured neighborhood of c_0

$$q: \quad K := \operatorname{Spec} k((t)) \cong \operatorname{Spec} \operatorname{Frac} \hat{\mathcal{O}}_{C,c_0} \longrightarrow \operatorname{Spec} \operatorname{Frac} \mathcal{O}_{C,c_0} \longrightarrow G$$

s.t. $\lim_{t \to 0} [q(t) \cdot z] = 0$. $\quad \hookrightarrow$ K-valued pt of G

- Step 4:

  - $K := \operatorname{Spec} k((t)) \longrightarrow R := \operatorname{Spec} k[[t]]$ natural morphism

$\Rightarrow$ R-valued pts of G form a subgroup of the K-valued pts
i.e., $G(R) \subset G(K)$

  - $\operatorname{Spec} k \to R$ induces $G(R) \longrightarrow G(k)$ given by taking specialization $t \to 0$.

  - $K \longrightarrow \mathbb{G}_m = \operatorname{Spec} k[s, s^{-1}]$ induced by $k[s,s^{-1}] \to k((t))$
$s \longmapsto t$

for 1-PS λ of G define its Laurent series expansion

$$\langle \lambda \rangle \in G(K)$$
$$\shortparallel$$
$$\left(K \longrightarrow \mathbb{G}_m \xrightarrow{\lambda} G \right)$$

• Step 5:

Cartan-Iwahori (every double coset in $G(K)$ for the subgroup $G(R)$ is represented by a Laurent series expansion $\langle\lambda\rangle$ of 1-PS of G)

$\Longrightarrow$ as $q \in G(K)$, $\exists \ell_1, \ell_2 \in G(R)$ and $\exists$ 1-PS λ of G s.t.

$$\ell_1 \cdot q = \langle\lambda\rangle \cdot \ell_2$$

q not R-valued point of $G \Longrightarrow \lambda$ non-trivial.

• Step 6:

let $g_i := \ell_i(0) \in G$. Then

$$0 = g_1 \cdot 0 = \lim_{t\to 0} \ell_1(t) \cdot \lim_{t\to 0} (q(t)\cdot z) =$$

$$= \lim_{t\to 0} \left[(\langle\lambda\rangle \cdot \ell_2)(t) \cdot z \right] \qquad [1]$$

$\left.\begin{array}{l} \ell_2 \in G(R) \\ g_2 = \lim_{t\to 0} \ell_2(t) \end{array}\right| \mapsto \ell_2(t)\cdot z = g_2 \cdot z + \epsilon(t) = \sum_{r\in\mathbb{Z}} (g_2\cdot z)_r + \epsilon(t)_r \quad [2]$

only positive powers of t

Action of 1-PS λ on $V = \mathbb{A}^n$ decomposes into weight spaces V_r, $r \in \mathbb{Z}$.

$[1] \& [2] \Rightarrow (g_2 \cdot z)_r = 0 \;\; \forall r \leq 0 \Rightarrow \lim_{t\to 0} \lambda(t) \cdot g_2 \cdot z = 0 \Longrightarrow$

$\Longrightarrow \lambda' := g_2^{-1} \lambda g_2$ is a 1-PS on G with $\lim_{t\to 0} \lambda'(t)\cdot z = 0$.

$\square$