GIT Talk 7 – Moduli Spaces of Vector Bundles

Plan for today & next week

* Vector bundle recap (locally-free & torsion free sheaves notion of rank & degree)
* Slope-(semi)-stability (slope, properties of homomorphisms of (s)st bundles)
* Quot schemes HARD!
* $\mathrm{VecBun}_{r,d}(X)^{sst} \cong$ subscheme of Quot scheme $/\!\!/ GL(r,k)$
* Slope-(s)st = GIT-(s)st HARD!

Extra topics:

* Harder-Narasimhan and Jordan-Hölder filtrations (& polystability, S-equivalence, Shatz polygons)
* Analytical approach using connections $\rightsquigarrow$ Deformation theory, Kempf-Ness

Vector bundle recap

X = smooth proj. conn. alg. curve $/\mathbb{C}$ (Riemann surface)

An algebraic vector bundle $p: E \longrightarrow X$ is a morphism s.t.

(i) $E_x = p^{-1}(x)$ is $\mathbb{C}$-VS

(ii) $\forall x \in X$ $\exists U \subset X$ open nbhd of x, Φ_U iso s.t.

$$\begin{tikzcd} E_U = p^{-1}(U) \arrow[rr, "\cong", "\Phi_U"'] \arrow[dr] & & U \times \mathbb{C}^{r(x)} \arrow[dl, "pr_2"] \\ & U & \end{tikzcd}$$

Φ_U is a $\mathbb{C}$-VS iso on fibers

* $r(x) = r \in \mathbb{N}$ rank of E
* Morphisms $\begin{tikzcd} E \arrow[rr] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd}$ $\rightsquigarrow$ $\mathrm{VecBun}(X)$ category (not abelian)

Characterization as Sheaves

$$\Gamma_E(U) = \mathcal{O}_X(U)\text{-module of sections } \sigma : U \to E$$

$$\Gamma : \mathrm{VecBun}(X) \xrightarrow{\cong} \mathrm{Loc\,Free}\ \mathcal{O}_X\text{-Mod}$$

Category equivalence

$\rightsquigarrow$ Sheaf theory applicable! (Sheaf Cohom, additivity of χ, Riemann-Roch, etc.)

Characterization through transition functions

Find cov. $(U_i)_{i\in I}$ with local triv.

$$\Phi_{U_j} \circ \Phi_{U_i}^{-1} : U_i \cap U_j \times \mathbb{C}^r \longrightarrow U_i \cap U_j \times \mathbb{C}^r$$

$$(x, v) \longmapsto (x, \sigma_{ij}(v))$$

1-cocycles, characterizes E up to isom.

$$[\sigma_{ij}] \in \underbrace{\check{H}^1(X, \mathcal{O}_X(GL(r,\mathbb{C})))}_{\text{a-priori only a set!!}} = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, GL(r,\mathbb{C}))$$

$\mathcal{O}_X(GL(r,\mathbb{C}))$ (not abelian unless $r = 1$)

Line bundles $(r=1)$ $\quad \mathcal{K}_X(U) = \{f : U \dashrightarrow \mathbb{C}\}$ $\quad \mathcal{K}_X(U)^*$ = nonzero rational functions

$$1 \to \mathcal{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{K}_X^*/\mathcal{O}_X^* \longrightarrow 1$$

$$\ldots \to \mathcal{K}_X^*(X) \to \left(\mathcal{K}_X^*/\mathcal{O}_X^*\right)(X) = \underbrace{Ca(X)}_{\substack{\text{Cartier}\\ \text{divisors}\\ (\mathcal{K}_X^*/\mathcal{O}_X^*)(X)}} \xrightarrow{\text{div}} H^1(X, \mathcal{O}_X^*) = \underbrace{Pic(X)}_{\substack{\text{Picard}\\ \text{Group}\\ \text{(of line bundles}\\ \text{on } X)}} \to \underbrace{H^1(X, \mathcal{K}_X^*)}_{=0} \ldots$$

$im(div) = Ca_p(X)$ principal divisors

<u>Claim:</u> $Pic(X) := H^1(X, \mathcal{O}_X^*) \cong Ca(X) / \underbrace{Ca_p(X)}_{}$ Linear equivalence

<u>Proof:</u> (X irreducible & $\mathcal{K}_X^*$ flabby (since constant) $\Rightarrow$ acyclic)

Def degree : $\mathrm{Pic}(X) \cong \mathrm{Ca}(X)/\mathrm{Ca}_p(X) \longrightarrow \mathbb{Z}$,

evaluate multiplicity at every $x \in X$ using valuation from $\mathcal{O}_{X,x}$ → sum to get $\deg(L) \in \mathbb{Z}$

For any rank: $\deg(E) = \deg(\det(E))$

Facts additive in SES ⇝

$$\deg(E \oplus F) = \deg(E) + \deg(F)$$
$$\deg(E \otimes F) = rk(F)\deg(E) + rk(E)\deg(F)$$
$$\deg(E^*) = -\deg(E)$$
$$\deg(E) = \int_X ch_1(E)$$

Def slope $\mu(E) = \deg(E)/rk(E)$

Facts Slope preserves slopes !

$$0 \to F \underset{\leq \atop \geq}{\overset{=}{\longrightarrow}} E \underset{\leq \atop \geq}{\overset{=}{\longrightarrow}} Q \to 0$$

Slope-(semi)-stability

Def E is slope-(semi)-stable if $\forall F \subset E$ subbundle: $\mu(F) \overset{(\le)}{<} \mu(E)$

Facts Line bundles stable, E (s)st $\Rightarrow L \otimes E$ (s)st, $\underset{sst}{E} \xrightarrow{\neq 0} \underset{sst}{F} \Rightarrow \mu(E) \le \mu(F)$

In SES of v.b. of equal slope, $0 \to \underset{sst}{F} \to E \to \underset{sst}{Q} \to 0 \Rightarrow E$ sst

E st $\Rightarrow \mathrm{Aut}(E) \cong \mathbb{C}^*$

Question: How to construct $\mathrm{VecBun}_{r,d}(X)^{sst}$?

$g \ge 2$, E rank r, deg d, $\underline{sst}$, such that $d > r(2g-1)$

(so $\deg(K) > 0$)

if not fulfilled by E, tensor with a line bundle so its true, (doesn't affect semistability)

Lemma

(i) $H^1(X, E) = 0$,

(ii) E globally generated, i.e. $\mathcal{O}_X^I \twoheadrightarrow E$

proof: (i) Assume $0 \neq H^1(X,E) \underset{\text{S.D.}}{\cong} H^0(X, E^* \otimes K)^*$, $0 \to \ker \to E \xrightarrow{f \neq 0} K \longrightarrow 0$

$\mu(\ker) > \frac{d}{r} > (2g-2)$

$\lightning$ sst of E.

(ii) $\forall x \in X$: E_x torsion sheaf on X, SES:

$$0 \longrightarrow \mathcal{O}_X(-x) \otimes E \longrightarrow E \longrightarrow E_x = E \otimes \mathbb{C}_x \longrightarrow 0$$

Sheaf Coh:

$$0 \to H^0(X, \mathcal{O}_X(-x) \otimes E) \to \underbrace{H^0(X,E) \to H^0(X, E_x)}_{\text{surjective}} \to \underbrace{\overset{=0}{H^1}}_{\substack{\text{Vanishes through} \\ \text{same argument} \\ \text{as in (i)}}} \cdots$$

(Tensoring does not affect semistability)

$\forall x \in X$

$\Rightarrow H^0(X,E) \otimes \mathcal{O}_X \to E$ surjective

Quot Schemes

Def $\mathcal{F}$ coh $\mathcal{O}_X$-Sheaf, family $\mathcal{E}$ of quotients of $\mathcal{F}$ (over C)
is (i) $\mathcal{O}_{X\times S}$-module over $X\times C$
(ii) $\sigma: pr_2^*\mathcal{F} \longrightarrow \mathcal{E}$ is surjective map of $\mathcal{O}_{X\times C}$-Modules
s.t. $\mathcal{E}$ is flat over C

$(\mathcal{E},\sigma)\cong(\mathcal{E}',\sigma')$ if $\mathrm{Ker}(\sigma)\cong\mathrm{Ker}(\sigma')$

(compatible with pullbacks) $\rightsquigarrow$ $\mathrm{Quot}(\mathcal{F}): \mathrm{Sch}^{op}\longrightarrow \mathrm{Sets}$ (Moduli functor Representable by Scheme?)

<u>Idea</u>: Fix ample line bundle L over X, $\mathcal{E}_c$ coh sheaf on X, pullback of $X\times\{c\}\longrightarrow X\times C$

$P_{L,\mathcal{E}_c}(t) = \chi(\mathcal{E}_c\otimes L^t)$ polynomial for $t\gg 0$ (Serre-Vanishing)

$\rightsquigarrow \mathrm{Quot}^{P,L}(\mathcal{F})$ This is representable by Scheme!

Sidenote: $\mathrm{Quot}^{P,L}(\mathcal{O}_X) = \mathrm{Hilb}^{P,L}(X)$, use ideal sheaves

(Tudor used Hilb poly of ideal sheaves)

For us: $\mathcal{F} = E$ vector bundle over X rank r deg d, $\deg(L) = 1$

$$P_{L,E}(t) = \chi(E \otimes L^t) = \underbrace{d + rt}_{\deg(E \otimes L^t)} + rt + r(1-g) \quad \text{(Riemann-Roch)}$$

possible to interchange P with (r,d)

$$\rightsquigarrow \mathrm{Quot}^{r,d}(\mathcal{F})$$

Using lemma: $g \geq 2$, $d > r(2g-1)$, E (r,d) sst., $\underbrace{H^0(X,E) \cong \mathbb{C}^N}_{\text{fixed}}$

then $\mathcal{O}_X^N \twoheadrightarrow E$, so $\mathrm{VecBun}_{r,d}(X)^{sst} \subset \mathrm{Quot}^{r,d}(\mathcal{O}_X^N)$?

Theorem (8.43 in Hoskins) $\mathrm{Quot}_X^{P,L}(\mathcal{F})$ can be represented by a projective scheme.

"proof" Step 1 Reduce to $X = \mathbb{P}^n$, $L = \mathcal{O}(1)$, $\mathcal{F} = \mathcal{O}^N$

WLOG L very ample $\rightsquigarrow$ $X \overset{i}{\hookrightarrow} \mathbb{P}^n$ embedding $L = i^*\mathcal{O}(1)$

i closed imm. $\Rightarrow$ $i_*\mathcal{F}$ coherent, i_* exact induces

$$\mathrm{Quot}(\mathcal{F}) \Longrightarrow \mathrm{Quot}(i_*\mathcal{F}) \text{ nat trans}$$

"closed immersion"

Step 2 $\mathrm{Quot} \hookrightarrow \mathrm{Gr}(\mathbb{C}^N \otimes \mathcal{O}(m)(\mathbb{P}^n))$

Similar to Uniform-m-Lemma from Tudor's talk, here: Castelnuovo-Mumford Reg.

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbb{P}^N}^n \to \mathcal{F} \to 0$$

(1) $H^i(\mathcal{K}(m))$, $H^i(\mathcal{F}(m))$ vanish for $m >> 0$

(2) $\mathcal{K}(m)$, $\mathcal{F}(m)$ globally generated, i.e. $\mathcal{O}_X^I \twoheadrightarrow \mathcal{K}(m)$) (induction on n).

Step 3 Quot is represented by a locally closed subscheme of Gr

$\mathrm{Quot}(\mathcal{F}) \Longrightarrow \mathrm{Quot}(i_*\mathcal{F})$

"locally closed immersion"

Step 4 Quot proper (Valuative crit. for properness)

(1) Quot is quasi-projective, using Plücker embedding

$$\mathrm{Gr}(W, P(m)) \hookrightarrow \mathbb{P}(\Lambda^{P(m)} W^*) \rightsquigarrow$$ Quot separated, finite type

(2) Show: DVR R, $\mathrm{Frac}(R) = K$, (over $\mathbb{C}$)

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathrm{Quot} \arrow[d, equal] \\ \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed, "\exists!"] & \mathrm{Quot} \end{tikzcd}$$

$\{K\text{-points}\} \longleftrightarrow \{R\text{-points}\}$

$\longleftarrow$ injective, since separated

GIT construction of $\mathrm{VecBun}_{r,d}(X)^{sst}$

$R^{(s)st}_{r,d}$ = (semi)-stable locus in $\mathrm{Quot}^{r,d}(\mathcal{O}_X^N) = Q$
open subscheme

Construction There exists a GL_N action on Q preserving $R^{(s)st}_{r,d}$, such that $R^{(s)st}_{r,d} / GL_N = \{ \mathrm{VecBun}_{r,d}(X)^{sst} \}$ as sets.

$$g \cdot (\mathcal{O}_X^N \twoheadrightarrow E) = (\mathcal{O}_X^N \xrightarrow{g^{-1}} \mathcal{O}_X^N \xrightarrow{q} E).$$ This action clearly preserves $R^{(s)st}$

Through $Q \hookrightarrow Gr(\mathbb{C}^N \otimes \mathcal{O}(m)(\mathbb{P}^n)) \underset{\text{Plücker}}{\hookrightarrow} \mathbb{P}$, we find a linearization L (pull back $\mathcal{O}(1)$ on $\mathbb{P}$ to Q) and then construct the projective GIT-quotient $R^{(s)st}_{r,d} /\!/_L GL_N$

.... To be continued!