Recap of projective GIT (base field $k = \mathbb{C}$)

$X \hookrightarrow \mathbb{P}^n$ closed subscheme, i.e. $X = \mathrm{Proj}\Big(\underbrace{\mathbb{C}[x_0, \dots, x_n] / I}_{R(X)}\Big)$

G reductive group

$G \curvearrowright \mathbb{P}^n$ linearly preserving X

Then: the projective GIT quotient is $X /\!/ G := \mathrm{Proj}(R(X)^G)$.

$$X \dashrightarrow X/\!/G \qquad X^{ss} \longrightarrow X/\!/G$$

Set-theoretically: $X/\!/G$ is the collection of polystable orbits.

- If X is a smooth projective variety (i.e. a complex submanifold of $\mathbb{P}^N$), there is also an analytical approach to these quotients.
  - $K \subset G$ maximal compact subgroup $(G = \mathbb{C}^*, K = S^1)$
  - X inherits a symplectic form from $\mathbb{P}^n$
  - the action $K \curvearrowright X$ is Hamiltonian with moment map $\mu : X \to k^*$

  Then the symplectic quotient is $X/\!/K := \mu^{-1}(0)/K$.

  Theorem: [Kempf-Ness] GIT quotient = symplectic quotient

- ~~Starting point: $\mathbb{CP}^n = \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^*$~~ $\rightsquigarrow$ our favourite "Kähler manifold"

- Starting point: $\mathbb{C}^n \cong \mathbb{R}^{2n}$ $(z_1, \dots, z_n)$ $(x_1, y_1, \dots, x_n, y_n)$. $z_j = x_j + i y_j$

Hermitian form: $H_0 : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$, $H(z, w) = \bar{z}_1 w_1 + \dots + \bar{z}_n w_n$

Breaks into real and imaginary parts: $z = x + iy$. $w = x' + iy'$

$$H_0(z,w) = \underbrace{(x_1 x_1' + y_1 y_1' + \dots + x_n x_n' + y_n y_n')}_{\text{inner product } g_0(z,w)} + i \underbrace{(x_1 y_1' - x_1' y_1 + \dots + x_n y_n' - x_n' y_n)}_{\text{symplectic form } \omega_0(z,w)}$$

More generally: $V \cong \mathbb{R}^{2n}$ real v.s. of dim. $2n$

$\hookrightarrow$ 3 kinds of structures we may have

① Complex structure. $J : V \to V$ linear s.t. $J^2 = -1$

In $\mathbb{C}^n$: $J_0(z_1, \dots, z_n) = (iz_1, \dots, iz_n)$; $J_0(x_1, y_1, \dots, x_n, y_n) = (-y_1, x_1, \dots, -y_n, x_n)$

② Inner product, $g : V \times V \longrightarrow \mathbb{R}$ bilinear which is symmetric $g(u,v) = g(v,u)$
positive definite $\forall u \neq 0$ $g(u,u) > 0$

③ Symplectic form: $\omega : V \times V \to \mathbb{R}$ bilinear which is skew-symmetric $\omega(u,v) = -\omega(v,u)$
non-degenerate: if for some $u \in V$ we have $\forall v \in V$ $\omega(u,v) = 0$, then $u = 0$

$$\tilde{g} : V \to V^*, \quad u \mapsto g(u, \cdot) \qquad \tilde{\omega} : V \to V^*, \quad u \mapsto \omega(u, \cdot)$$

If these structures are "compatible", then they combine to give a Hermitian form $H : V \times V \to \mathbb{C}$
- H is complex linear in the second argument $H(u, Jv) = i H(u,v)$
- $H(u,v) = \overline{H(v,u)}$
- positive-definite

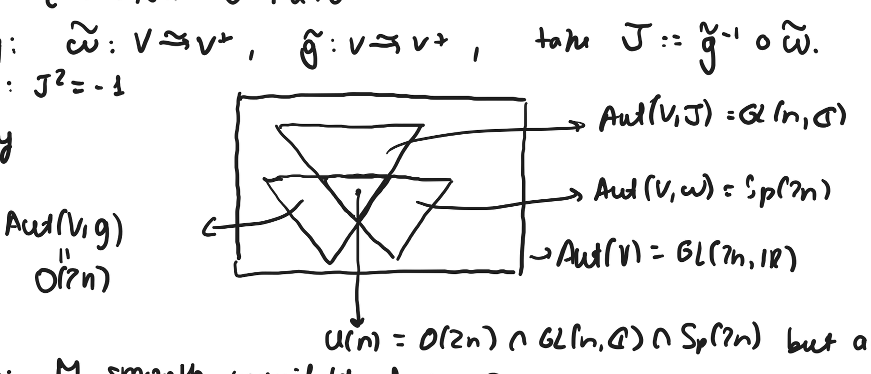
$H(u,v) := g(u,v) + i\,\omega(u,v)$

Compatibility condition: $\omega(u, Jv) = g(u,v) \iff g(Ju, v) = \omega(u,v)$

other things that are true: $g(Ju, Jv) = g(u,v)$, $\omega(Ju, Jv) = \omega(u,v)$

Note: Every 2 out of 3 of (J, g, ω) determines the third one!
- given J, g s.t. $g(Ju, Jv) = g(u,v)$, then $\omega(u,v) := g(Ju, v)$.
- given J, ω s.t. $\begin{cases} \omega(Ju, Jv) = \omega(u,v), \\ \omega(u, Ju) > 0 \ \forall u \neq 0 \end{cases}$ then $g(u,v) := \omega(u, Jv)$.
- given ω and g: $\tilde{\omega} : V \xrightarrow{\sim} V^*$, $\tilde{g} : V \xrightarrow{\sim} V^*$, take $J := \tilde{g}^{-1} \circ \tilde{\omega}$.
  compatibility: $J^2 = -1$

2-out-of-3 property



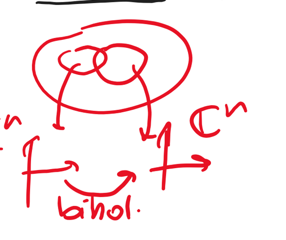
$U(n) = O(2n) \cap GL(n, \mathbb{C}) \cap Sp(2n)$ but also the pairwise intersections!

Moving to manifolds: M smooth manifold of dim. $2n$

Strategy: impose linear structure at each point. smoothly [but sometimes we need extra conditions]

① Almost complex structure: $J_p : T_pM \to T_pM$, $J_p^2 = -1$

Remark: If M is a complex manifold $\dim_{\mathbb{C}} = n$ $(\dim_{\mathbb{R}} = 2n)$, it has a canonical a.c.s.



in local complex coordinates $(z_1, \dots, z_n)$ on U, $p \in U$

$(x_1, y_1, \dots, x_n, y_n)$

$T_pM := \mathrm{span}_{\mathbb{R}}\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right\}$ $\quad J_0 : T_pM \to T_pM$ $\quad \frac{\partial}{\partial x_j} \mapsto \frac{\partial}{\partial y_j}$, $\frac{\partial}{\partial y_j} \mapsto -\frac{\partial}{\partial x_j}$.

- Not every a.c.s. comes from a complex manifolds; those are called integrable.

② Riemannian metric: $g_p : T_pM \times T_pM \to \mathbb{R}$
No integrability condition

③ Almost symplectic structure $\omega_p : T_pM \times T_pM \to \mathbb{R}$ $\quad \omega \in \Omega^2(M)$

Integrability: $d\omega = 0$
- $[\omega] \in H^2(M; \mathbb{R})$
- Darboux's theorem: (M, ω) is locally symplectomorphic to the std. $(\mathbb{R}^{2n}, \omega_0)$

Bring everything together: Def: A Kähler manifold (M, g, J, ω) of compatible structures. $g(J\cdot, \cdot) = \omega(\cdot, \cdot)$

These can be combined to give a Hermitian metric $H = g + i\omega$ $\quad H_p : T_pM \times T_pM \to \mathbb{C}$

Time for our lovely $\mathbb{CP}^n$: $M = \mathbb{CP}^n = \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^*$ is a complex manifold

$T_{[z]}\mathbb{CP}^n = ?$ $\quad \pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{CP}^n$ submersion

$d\pi_z : \mathbb{C}^{n+1} \twoheadrightarrow T_{[z]}\mathbb{CP}^n$ $\quad \ker d\pi_z = \mathbb{C} \cdot z$

$(\mathbb{C} \cdot z)^{\perp} \cong \mathbb{C}^{n+1} / \mathbb{C} \cdot z \xrightarrow{\cong} T_{[z]}\mathbb{CP}^n$

Problem: H is not $\mathbb{C}^*$-invariant $\quad H(\lambda z, \lambda w) = |\lambda|^2 H(z,w)$ ✗ $\quad \mathbb{CP}^n = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\mathbb{C}^*} \cong \frac{S^{2n+1}}{S^1}$

$\hookrightarrow$ it is S^1-invariant

Now it works: $H_{FS} = g_{FS} + i\,\omega_{FS}$ on $\mathbb{CP}^n$ $\quad$ FS — Fubini-Study

$$\begin{array}{ccc} S^{2n+1} & \overset{i}{\hookrightarrow} & \mathbb{C}^{n+1} \\ \pi \downarrow & & \\ \mathbb{CP}^n & & \end{array}$$

- ω_{FS} is uniquely defined by $\pi^* \omega_{FS} = i^* \omega_0$
- g_{FS} is the quotient metric induced by $i^* g_0$

$$\underset{\pi^* d\omega_{FS}}{\overset{\|}{d(\pi^*\omega_{FS})}} = d(i^*\omega_0) = i^* d\omega_0 = 0 \xrightarrow{\pi^* \text{ injective}} d\omega_{FS} = 0 \qquad \therefore \mathbb{CP}^n \text{ is Kähler}$$

$\hookrightarrow$ ω_{FS} is $U(n+1)$-invariant (this also determines ω_{FS} up to scaling)

- M Kähler manifold. $X \overset{i}{\subseteq} M$ complex submanifold $\longrightarrow$ X is Kähler with $i^*\omega$ Kähler form

Corollary: Every smooth projective variety embedded in $\mathbb{P}^N$ inherits a canonical Kähler structure!

More on symplectic stuff (M, ω) symplectic manifold

- Hamiltonian dynamics: $H : M \to \mathbb{R}$ smooth function

The Hamiltonian vector field associated to H is X_H defined by $\omega(X_H, \cdot) = dH$.
The flow of X_H $\{\phi_t^H\}_{t \in \mathbb{R}}$ is a 1-parameter family of Hamiltonian diffeomorphisms

Prop: Hamiltonian flows preserve H, ω. [Cartan magic formula]
$\mathcal{L}_{X_H} H = 0 \qquad \mathcal{L}_{X_H} \omega = 0$

All Hamiltonian diffeos are symplectic, but not all symplectic diffeos. are Hamiltonian.
(obstruction: $H^1(M; \mathbb{R})$)

- Group actions on symplectic manifolds: K compact Lie group, $K \curvearrowright M$

$\to$ The action is symplectic if K acts by symplectic diffeos.

We want to require more: that K acts by Hamiltonian diffeos $\curvearrowright$ we look at the infinitesimal generators of the action

$\mathfrak{k} = \mathrm{Lie}\,K$. $\xi \in \mathfrak{k}$ generate a 1-p subgroup of K $\{\exp t\xi : t \in \mathbb{R}\}$
(where $\exp : \mathfrak{k} \to K$)

The infinitesimal generator associated to ξ is $X_\xi \in \mathfrak{X}(M)$ defined by $X_\xi(p) := \frac{d}{dt}\left[\exp(t\xi) \cdot p\right]\Big|_{t=0}$.

We require that each X_ξ is a Hamiltonian vector field.

If we choose such Hamiltonians for each $\xi \in \mathfrak{k}$ and collect them: $\mathfrak{k} \to C^\infty(M)$ $\quad M \to \mathfrak{k}^*$
$\mathfrak{k} \to (M \to \mathbb{R}) \rightsquigarrow M \to (\mathfrak{k} \to \mathbb{R})$

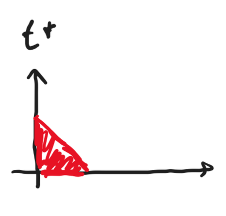
We want the choices to be "consistent": we require
$\mu : M \to \mathfrak{k}^*$ to be K-equivariant.
$\circlearrowleft_K \qquad \circlearrowleft_K$ coadjoint action

Def: A symplectic action $K \curvearrowright M$ is Hamiltonian if there exists a $\mu : M \to \mathfrak{k}^*$ s.t. (moment map)
- μ is K-equivariant
- for each $\xi \in \mathfrak{k}$, the function $\mu_\xi = \langle \mu(\cdot), \xi \rangle$ is a Hamiltonian for X_ξ.
  $\langle \cdot, \cdot \rangle : \mathfrak{k}^* \times \mathfrak{k} \to \mathbb{R}$

Sidenote: $K \curvearrowright K$ by conjugation $\quad g \cdot h = ghg^{-1}$
$K \curvearrowright \mathfrak{k}$ by differentiating (adjoint action)
$K \curvearrowright \mathfrak{k}^*$ dual (coadjoint action)

- The moment map is in general non-unique: you can add a constant in $[\mathfrak{k}, \mathfrak{k}]^\circ$

Example: $\mathbb{T}^2 \curvearrowright \mathbb{CP}^2$ $\quad (t_1, t_2) \cdot [z_0 : z_1 : z_2] = [z_0 : t_1 z_1 : t_2 z_2]$
$\mu : \mathbb{CP}^2 \to \mathbb{R}^2$ $\quad \mu([z_0 : z_1 : z_2]) = \left(\frac{|z_1|^2}{|z|^2}, \frac{|z_2|^2}{|z|^2}\right)$



- $U(n+1) \curvearrowright \mathbb{CP}^n$ is Hamiltonian

Symplectic reduction: (M, ω) symplectic manifold, G compact Lie group

$G \curvearrowright M$ Hamiltonian action with moment map $\mu : M \to \mathfrak{g}^*$
G acts freely on $\mu^{-1}(0)$ (the stabilizers are trivial)

$$\begin{array}{ccc} \mu^{-1}(0) & \overset{i}{\hookrightarrow} & M \\ \pi \downarrow & & \\ M/\!/G & & \end{array}$$

Then:
- $\mu^{-1}(0)$ is a submanifold of M
- $M_{red} = M/\!/G := \mu^{-1}(0)/G$ is a smooth manifold
- $\pi : \mu^{-1}(0) \to M/\!/G$ is a principal G-bundle
- there is a unique symplectic form ω_{red} on M_{red} s.t. $\pi^*\omega_{red} = i^*\omega$.

$\dim M/\!/G = \dim M - 2 \dim G$

Example: $M = \mathbb{C}^{n+1}$, $G = S^1$ $\quad S^1 \curvearrowright \mathbb{C}^{n+1}$ $\lambda \cdot \vec{z} = \lambda \vec{z}$
$\mu^{-1}(0) = S^{2n+1}$ $\quad M/\!/G = S^{2n+1}/S^1 \cong \mathbb{CP}^n$

(one possible choice is)
$\mu : \mathbb{C}^{n+1} \to \mathbb{R}$
$\mu(z) \sim \frac{1}{2} - \frac{1}{2}|z|^2$
[We identify $\mathrm{Lie}(S^1) \cong \mathbb{R}$ such that $\exp : \mathbb{R} \to S^1$, $t \mapsto e^{it}$.]

- If G is not free:
  - if the stabilizers are finite: we get an orbifold
  - more generally: stratified symplectic space

- This also can be made to work in infinite dimensions.