Moduli space of vector bundles II [Ander]

Remark: $\mathrm{Quot}_Y^{P,L}(F)$ is a fine moduli space!
the identity morphism

GIT setup for the construction of the moduli space:

Let X be a nice curve. Assume $g \geq 2$. for Riemann-Roch to work

Fix n (the rank) and choose d such that $d > n(2g-1)$
we assume it to be large to have many global sections

Let $\mathcal{E}$ be a locally free sheaf of rank n.
Serre duality: $H^i(X,\mathcal{E}) \cong H^{1-i}(X, K_X \otimes \mathcal{E}^\vee)^\vee$

(curves) $\dim H^0(X,\mathcal{E}) = d + n(1-g) =: N$ we consider $\mathcal{O}_X^N$ as the sheaf to define the Quot scheme!

$H^0(X,\mathcal{E}) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{E}$ take enough sections

$q: \mathcal{O}_X^N \longrightarrow \mathcal{E}$ surjection

$Q := \mathrm{Quot}_X^{n,d}(\mathcal{O}_X^N)$ d degree, n rank

Let $R^{(s)s} \subset Q$ the open subscheme consisting of (semi)stable locally free sheaves of rank n

$$q_Q: \mathcal{O}^N_{Q\times X} \twoheadrightarrow \mathcal{U}, \quad \mathcal{U}^{(s)s} := \mathcal{U}|_{R^{(s)s}}$$

$$q^{(s)s}: \mathcal{O}^N_{R^{(s)s}\times X} \twoheadrightarrow \mathcal{U}^{ss}$$

Lemma 8.48: The universal quotient sheaf $\mathcal{U}^{(s)s}$ over $R^{(s)s} \times X$ is a family over $R^{(s)s}$ of (semi)stable locally free sheaves over X with invariants (n, d) with the local universal property.

Lemma 8.49: $\exists\, GL_N \curvearrowright Q$ s.t. the orbits in $R^{(s)s}$ are in 1:1 correspondence with semistable locally free sheaves on X with invariants (n, d) (up to isomorphism)

$$\mathcal{O}_X(X) = k, \quad \mathcal{O}_X^N(X) = k^N$$

$$\sigma : GL_N \times Q \to Q$$

$$g(\mathcal{O}_X^N \xrightarrow{q} \mathcal{E}) = (\mathcal{O}_X^N \xrightarrow{g^{-1}} \mathcal{O}_X^N \to \mathcal{E})$$

We want to linearise this action. We want to find a line bundle on X s.t.

$$\pi : \begin{cases} L \to X & \text{is } G\text{-equivariant} \\ L_x \to L_{g \cdot x} & \text{is linear} \end{cases}$$

$$L \in \mathrm{Pic}(X) \rightsquigarrow \mathbb{L} = \underline{\mathrm{Spec}}_X(\mathrm{Sym}^\bullet L^\vee)$$

<u>Slogan:</u> "Quot generalises Grassmannians"

$$Q = \mathrm{Quot}_X^{n,d}(\mathcal{O}_X^N) \hookrightarrow \mathrm{Gr}(H^0(\mathcal{O}_X(m)), M) \text{ for } M = mn + d + n(1-g)$$
$$\downarrow$$
$$\mathbb{P}(\Lambda^M H^0(\mathcal{O}_X^N(m))^\vee)$$

$Q \xhookrightarrow{\iota} \mathbb{P}$, consider $\iota^* \mathcal{O}(1) =: \mathcal{L}_m$

We have a linear action of SL_n on $H^0(\mathcal{O}_X^N(m)) \rightsquigarrow \mathcal{L}_m$ admits a linearisation

<u>Explicitly:</u> $\mathcal{L}_m = \det(\pi_{Q,*}(\mathcal{U} \otimes \pi_X^* \mathcal{O}_X(m)))$

$$\begin{array}{ccc} Q \times X & \xrightarrow{\pi_X} & X \\ {\scriptstyle \pi_Q}\downarrow & & \\ Q & & \end{array}$$

$\Lambda^{\mathrm{rank}} \mathcal{E}$ is a line bundle

Recall: $G \times X \xrightarrow{\sigma} X$

$L \in \mathrm{Pic}(X)$. $\pi_X^* L \simeq G \times L \simeq \sigma^* L$

$$\overset{\circlearrowright SL_N}{H^0(\mathcal{O}_X^N(m))} = k^N \otimes H^0(\mathcal{O}_X(m))$$

$\mathcal{U}$ admits a SL_N-linearisation (the following maps are equivalent)

$$k^N \otimes \mathcal{O}_{SL_N \times Q \times X} \xrightarrow[\text{taking the inverse}]{(\sigma \times id_X)^* q_Q} (\sigma \times id_X)^* \mathcal{U}$$

$$\begin{array}{ccc} k^N \otimes \mathcal{O}_{SL_N \times Q \times X} & \xrightarrow{p_{SL_N}^* \tau} & k^N \otimes \mathcal{O}_{SL_N \times Q \times X} \\ & & \downarrow p_{Q \times X}^* q_Q \\ & & p_{Q \times X}^* \mathcal{U} \end{array}$$

induces an isomorphism

$\Phi : (\sigma \times id_X)^* \mathcal{U} \longrightarrow p_{Q \times X}^* \mathcal{U}$ satisfying the cocycle condition.

The stalks of $\mathcal{L}_m$ are, for $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ → a point $q \in Q$!!

$$\mathcal{L}_{m,q} \simeq \det H^*(X, \mathcal{F}(m)) = \bigotimes_{i \geq 0} \det H^i(X, \mathcal{F}(m))^{\otimes (-1)^i}$$

For $m >> 0$, $\mathcal{L}_{m,q} = \det H^0(X, \mathcal{F}(m))$

<u>Analysis of semistability</u> Let SL_N act on $Q = \mathrm{Quot}_X^{n,d}(\mathcal{O}_X^N)$ as above.

<u>Fact</u>: $Q^{ss}(\mathcal{L}_m) = R^{ss}$, $Q^s(\mathcal{L}_m) = R^s$

<u>Proposition 8.5.4</u> Let $q : \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ be a k-point in Q. Then $q \in Q^{(s)s}(\mathcal{L}_m)$ iff for all subspaces $0 \neq V' \subsetneq V = k^N$ we have an inequality $\frac{\dim V'}{p(\mathcal{F}', m)} \underset{(-)}{\leq} \frac{\dim V}{p(\mathcal{F}, m)}$, where $\mathcal{F}' := q(V' \otimes \mathcal{O}_X) \subset \mathcal{F}$.

Corollary 8.57 $\exists M$ s.t. for $m \geq M$ and a k-point $(q: \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}) \in \mathcal{Q}$ TFAE:

(1) q is (GIT)-(semi)stable for SL_N acting on $\mathcal{Q}$ wrt $\mathcal{L}_m$

(2) for all subschemes $F' \subset F$ and $V' := H^0(q)^{-1}(H^0(F')) \neq 0$, we have rank $F' > 0$ and $$\frac{\dim V'}{\operatorname{rk} F'} (\leq) \frac{\dim V}{\operatorname{rk} F}.$$

Proposition 8.61 Fix n, d s.t. $d > gn^2 + n(2g-2)$. Let $\mathcal{F}$ be a semistable locally free sheaf of rank n and degree d over X.

Then $\forall\, 0 \neq F' \subseteq F$ we have $$\frac{h^0(X, F')}{\operatorname{rk} F'} \leq \frac{h^0(X, \mathcal{F})}{\operatorname{rk} \mathcal{F}},$$

and if equality holds then $h^1(X, F') = 0$ and $\mu(F') = \mu(F)$.

Remark For $d \gg 0$, (semi)stability of $\mathcal{F}$ is equivalent to $$\frac{h^0(X, \mathcal{F}')}{\operatorname{rank} \mathcal{F}'} (\leq) \frac{h^0(X, \mathcal{F})}{\operatorname{rk} \mathcal{F}} \quad \forall\, 0 \neq \mathcal{F}' \subseteq \mathcal{F}.$$

Theorem 8.63: Let n, d be fixed s.t. $d > \max\{n^2(2g-2), gn^2 + n(2g-2)\}$.

Then $\exists M > 0$ s.t. $\forall m \geq M$ we have $\mathcal{Q}^{ss}(\mathcal{L}_m) = R^{ss}$,

$\mathcal{Q}^{s}(\mathcal{L}_m) = R^{s}$.

Construction of the moduli space

Theorem 8.64 There is a coarse moduli space $M^s(n,d)$ for moduli of stable vector bundles of rank n and degree d over X, and this has a natural projective completion $M^{ss}(n,d)$ whose k-points parametrize polystable vector bundles of rank n and degree d.

Proof (Sketch): Assume $d >> 0$ (as in previous result)

Let SL_N act on Q and let L_m be the linearising line bundle.

There is a projective GIT quotient

$$\pi: R^{ss} = Q^{ss}(\mathcal{L}_m) \longrightarrow Q /\!/_{L_m} SL_N =: M^{ss}(n,d)$$

which restricts to a geometric quotient

$$\pi: R^s = Q^s(L_m) \longrightarrow Q^s(L_m)/SL_N =: M^s(n,d)$$

$M^s(n,d)$ is a coarse moduli space.

It remains to show: $q: \mathcal{O}_X^N \twoheadrightarrow \mathcal{F}$ is closed $\Leftrightarrow$ $\mathcal{F}$ is polystable

$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ w/ $\mathcal{F}', \mathcal{F}''$ semistable,
non-split

$\mu(F') = \mu(F'') = \mu(F)$

$\exists$ 1-ps λ s.t. $\lim_{t \to 0} \lambda(t) \cdot [q] = [\mathcal{O}_X^N \longrightarrow \mathcal{F}'' \oplus \mathcal{F}']$

not closed $\Rightarrow$ orbit not closed (not polystable)

the other way also works (not polystable $\Rightarrow$ the seq. cannot split)

(Then it gets complicated and ugly, we won't do it)

Prop: $M^s(n,d)$ is smooth, quasi-projective of dim $n^2(g-1)+1$

Facts used in the proof

(1) $T_q Q \simeq \mathrm{Hom}(\ker q, F)$

(2) If $\mathrm{Ext}^1(\ker q, F) = 0 \rightsquigarrow Q$ is smooth in nbhd of q.

Remark: $T_{[E]} M^s(n,d) = \mathrm{Ext}^1(E,E)$

smoothness of $M^s(n,d)$ depends on $\mathrm{Ext}^2 = 0$. Since X is a curve ✓

- If $(n,d)=1$ then $M^s(n,d) = M^{ss}(n,d)$

Theorem: 8.68 If $(n,d)=1$, then $M^s(n,d) = M^{ss}(n,d)$ is a fine moduli space. smooth projective!