Moduli space of curves

[Noah Dizep]

- C smooth curve being stable $\Rightarrow$ Giesecher criterion

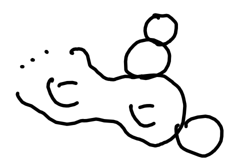
$\Rightarrow g \geq 2 \quad M_g$

$\Rightarrow$ show that to each suitable pluricanonical model of a DM curve there exists a corresponding Hilbert point which lies in the stable locus of H

$$M_g \overset{+\text{DM stable}}{\rightsquigarrow} \overline{M}_g$$

Def: C is a 1d scheme over k, C is Deligne-Mumford (stable) if:

1. projective, complete, proper over $k = \mathbb{C}$
2. only nodal singularities (ordinary double points)
3. finite $\mathrm{Aut}(C)$
   - $\Rightarrow g=0$, at least 3 nodal singularities
   - $\Rightarrow g=1$, at least 1 nodal singularities



DM stability
$\Longrightarrow$ Ensures that every family of $g \geq 2$ DM-stable curves C over a DVR as a unique stable limit.

$\Rightarrow$ smoothing over DVR:

Scheme C over Spec R, R is DVR, $\varphi : C \to \operatorname{Spec} R$

C_η (generic fibre) $\longrightarrow$ fraction field of R

smooth points $\mapsto$ generic point; nodal singularities $\mapsto$ closed point $m \triangleleft R$ (fibre C_0)

construct C such that
- C_η smooth
- C_0 is singular but admits deformation

- Potential stability theorem (PST)

genus $g \geq 0$, $N := d - g + 1$

V is N-dimensional vector space, H Hilbert scheme of $\mathbb{P}(V)$
$= \mathrm{Hilb}_{p,g}(\mathbb{P}(V))$, $p(m) = md - g + 1$

$X \to H$, X universal curve, $L \cong \mathcal{O}(1)$

C potentially stable: (1) C is non-degenerate (not contained in any hyperplane)
(2) C is DM-stable

(3) linear series embedding C is complete and non-special

$h^0(C, L) = N$, $h^1(C, L) = 0$

(4) If Y is a complete subcurve with g_Y, meeting the rest of C in k_Y points, then

$$\left| \deg_Y(L) - \frac{d}{g-1}\left(g_Y - 1 + \frac{k_Y}{2}\right) \right| \leq \frac{k_Y}{2}.$$

Theorem: $d > 9(g-1)$ or equivalently $\frac{d}{N} < \frac{8}{7}$

$\Rightarrow$ $\exists M$ depending only on d and g s.t. $m \geq M$ and C in $\mathbb{P}(V)$ is a connected curve with semistable m^{th} Hilbert point $[C]_m$, then C is potentially semistable.

• Stable curves in (low dimensions) can have arbitrarily bad singularities for large enough g !!!!

Proof: (1) C_{red} is non-degenerate

(2) Every component is generically reduced

(3) If an irreducible subcurve Y of C is not a rational normal curve, then $\deg_Y(L) \geq 4$.

$$\underset{\mathbb{P}^1}{[s:t]} \mapsto \underset{\mathbb{P}^d}{[s^d : s^{d-1}t : \cdots : t^d]} \quad \text{Veronese}$$

(4) If Y_{red} irreducible subcurve of C, then its normalisation $\psi : Y_{NS} \to Y$ is unramified

(5) Every singular point of C_{red} has multiplicity 2.

(6) Every double point of C_{red} is a node.

(7) $H^1(C_{red}, L) = \{0\}$

(8) L is reduced, so $H^1(C,L) = \{0\}$, $\quad V = H^0(C,L)$

(9) $\forall \underset{\text{subcurve}}{Y}$ of C and all components E of the normalisation Y_{ns},

either $\deg E \geq \underbrace{k_{E,Y}}_{\text{\# points where they meet}}$ or E is a rational normal curve with $\deg_E(L) = k_{E,Y} - 1$

(10) subcurve criterion holds.

UCH is open

stable curves?

$X \to U$ a family of nodal curves

$\omega = \omega_{X/U}$

relative dualising sheaf

$f: X \to k$, X smooth variety $\quad \omega_{X/k} \cong \Omega^n_{X/k} \quad (n = \dim X)$

$\omega_{X/Y} \in D^b Coh(X)$ in bad cases

$f: X \to Y \rightsquigarrow$ curve $g \geq 2$

$\omega_{X/Y}$ is a line bundle on X that restricts to $\omega_C = \Omega^1_C$ on ~~each~~ fibre $C = f^{-1}(y)$, $y \in Y$

→ pluricanonical locus

<u>Def:</u> J of ν-canonically embedded curves is a closed subscheme of U over which $\mathcal{O}_H(1)|_J \cong \omega_{X/U}^{\otimes \nu}$.

$\omega_C^{\otimes \nu}$, $\nu \geq 3$ $\qquad$ PST: $\frac{d}{N} \leq \frac{8}{7} \Rightarrow \frac{d}{N} := \frac{2\nu}{2\nu - 1}$

$\rightsquigarrow \nu \geq 5$

$\Rightarrow$ every curve whose Hilbert point lies in the ss. locus H^{ss} of H is <u>potentially stable</u>.

<u>Prop:</u> J^{ss} is closed in H^{ss}

<u>Corollary:</u> (1) C in $\mathbb{P}(V)$ whose Hilbert point lies in J^{ss} is DM-stable

(2) J^{ss} contains ν-canonically Hilbert point of every DM-stable curve of genus g

(3) $J^{ss} = J^s$: every curve whose Hilbert point lies in J^{ss} is Hilbert stable

$\Rightarrow \mathrm{Hilb}_{\nu(2g-2)+1-g,\, g}\left(\mathbb{P}^{\nu(2g-2)-1}(V)\right)$

$\overset{\text{corollary}}{\Longrightarrow}$ iso. classes of stable curves genus g correspond bijectively to GIT stable $PGL(V)$ orbits in J^{ss} $\quad \overline{M}_g := J^{ss} /\!\!/ PGL(V)$

$\Rightarrow \overline{M}_g$ is projective

$\overline{M}_g$ is irreducible, reduced over k algebraically closed

Biblio:

- Morrison: "GIT methods for moduli of stable curves"