§ Introduction

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Aim: define moduli spaces $\boxed{\mathcal{M}_g}$, $\overline{\mathcal{M}}_g$, $\mathcal{M}_{g,m}$, $\overline{\mathcal{M}}_{g,m}$, ...

moduli space of genus g curves

Idea: if C is a smooth, connected, comple curve of genus g over $\mathbb{C}$

C has a canonical sheaf ω_C (which in this case is just Ω^1_C) and it turns out that $\omega_C^{\otimes 3}$ is very ample if $g \geq 2$

$\rightsquigarrow \quad C \hookrightarrow \mathbb{P}^{5g-6}$

Theorem: if C, C' are canonically embedded in $\mathbb{P}^{5g-6}$, then $C \cong C'$ are $\cong$ iff their embeddings are projectively equivalent.

$\rightsquigarrow$ if K is the subscheme of $\mathbb{P}^{5g-6}$ containing all of the embeddings, then

$$\mathcal{M}_g := H_g /\!/ PGL(5g-5, \mathbb{C})$$

what is this scheme?

Build some scheme generating the subsets of $\mathbb{P}^{5g-6}$ cut out in this way, call it H_g

Aim: Define a scheme $\mathcal{H}_n$ which parametrises closed subschemes of $\mathbb{P}^n$

Def: The Hilbert functor $H_n : Sch/k^{op} \longrightarrow Set$ is given by

$$H_n(X) = \left\{ \begin{matrix} \text{closed} \\ \text{subschemes} \end{matrix} \; S \subseteq \mathbb{P}^n \times X \right\}.$$

$\Gamma(S \xrightarrow{\pi} X) \to$ family of closed subschemes of $\mathbb{P}^n$ param. by X

Aim: want H_n to be representable, i.e. find a space $\mathcal{H}_n$ s.t.

$$H_n(X) = Hom(X, \mathcal{H}_n)$$

this is a "fine moduli space" for the "moduli problem" H_n

DOES NOT EXIST!

- We require $H_n(X) = \left\{ \begin{array}{l}\text{closed}\\ \text{subschemes}\end{array} S \subseteq \mathbb{P}^n \times X : \pi : S \to X \begin{array}{l}\text{proper}\\ \text{flat}\end{array} \right\}$

↳ Then it exists, but is "too big" (i.e. many components)

<u>Next:</u> fix the degree and dimension of the subscheme we are parametrising. Concretely: fix a Hilbert polynomial.

<u>Definition:</u> Let $(X, \mathcal{O}_X)$ be a subscheme of $\mathbb{P}^n$, then there exists a unique polynomial h_X s.t. $h_X(n) = h^0(X, \mathcal{O}_X(n)) [= \dim_k S(X)_n]$. for n large enough ⊛

<u>Fact:</u> the leading term of $h_X(m)$ is $d\frac{m^s}{s!}$, where d = degree of X, $s = \dim X$

$H_{n,p}(X) = \left\{ \begin{array}{l}\text{closed}\\ \text{subschemes}\end{array} S \subseteq \mathbb{P}^n \times X : \pi : S \to X \begin{array}{l}\text{proper}\\ \text{flat}\end{array} \text{ and all of the fibres have Hilbert polynomial } p \right\}$

Proof of existence of $\mathcal{H}_{n,p}$

<u>Def:</u> Let $X \subseteq \mathbb{P}^n$ be a closed subscheme, $\mathcal{I}_X$ its ideal sheaf

Define $\mathcal{I}_X(m)$ to be $\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}^n}(m)$

(or equivalently,

$$0 \to \mathcal{I}_X(m) \to \mathcal{O}_{\mathbb{P}^n}(m) \to \mathcal{O}_X(m) \to 0$$

$$(\ker\phi \longrightarrow k[x_0, \dots, x_n] \xrightarrow{\phi} S(X))$$

⇝ induces a LES in cohomology

$$0 \to H^0(\mathbb{P}^n, \mathcal{I}_X(m)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) \to H^0(X, \mathcal{O}_X(m)) \to H^1(\mathbb{P}^n, \mathcal{I}_X(m)) \to \cdots$$

notice that $H^0(\mathbb{P}^n, \mathcal{I}_X(m)) \hookrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$

⊛ Thm: [Hilbert] $\chi_X(n) = h^0(X, \mathcal{O}_X(n)) - h^1 + h^2 - \cdots$ is a polynomial.
Also, for $n \gg 0$, $\mathcal{O}_X(n)$ has no higher cohomology (Serre).

So if somehow X were to be <u>completely</u> determined by $H^0(\mathbb{P}^n, \mathcal{I}_X(m))$ for some m, then X can be encoded as a point in some Grassmannian.

Want to assume that: $\dim H^0(\mathbb{P}^n, \mathcal{I}_X(m))$ is the same for all X with same Hilbert poly. p. Also that m can be chosen uniformly across all subschemes X (fix ambient space)

If $H^1(\mathbb{P}^n, \mathcal{I}_X(m)) = 0$ then

$$h^0(\mathbb{P}^n, \mathcal{I}_X(m)) = h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) - h^0(X, \mathcal{O}_X(m))$$
$$= h_{\mathbb{P}^n}(m) - h_X(m) \text{ for } m \text{ large enough}$$

$\hookrightarrow$ dimension stabilises!

<u>Fact:</u> (The uniform m lemma)

Fix a Hilbert polynomial P. Then $\exists m_P \in \mathbb{Z}$ s.t. $\forall k \geq m_P$ the following holds: (and for all $X \subseteq \mathbb{P}^n$ with $h_X = P$)

1) $\mathcal{I}_X(k)$ is generated by global sections

(can recover $\mathcal{I}_X(k)$ from $H^0(\mathbb{P}^n, \mathcal{I}_X(k))$)

2) $H^0(\mathbb{P}^n, \mathcal{I}_X(k)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \longrightarrow H^0(\mathbb{P}^n, \mathcal{I}_X(k+1))$ is surjective

$\mathcal{I}_X(\ell)$ for $\ell \geq k$; by Serre this recovers $\mathcal{I}_X$ itself

3) $h^i(\mathbb{P}^n, \mathcal{I}_X(k)) = 0$ for $i > 0$

makes LES calculation work, also $h_X(k) = h^0(X, \mathcal{O}_X(n))$

4) $h_{\mathbb{P}^n}(k) = h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$

(Quite technical!)

$$H^0(\mathbb{P}^n, \mathcal{I}_X(m_P)) \hookrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m_P))$$

$\rightsquigarrow$ $X \subseteq \mathbb{P}^n$ with Hilbert polynomial P defines a point

$$[X] \in Gr(h_{\mathbb{P}^n}(m_P) - P(m_P),\ h_{\mathbb{P}^n}(m_P))$$

All of these points cut out a subset $\mathcal{H}_{P,n} \subseteq Gr(\cdots)$.

~~Scheme structure on $\mathcal{H}_{P,n}$~~

$$\begin{tikzcd} S \subseteq \mathbb{P}^n \times X \arrow[r, "\pi_P"] \arrow[d, "\pi_X"] & \mathbb{P}^n \\ X & \end{tikzcd} \qquad \rightsquigarrow \text{ map } X \longrightarrow \mathcal{H}_{P,n}$$

(as sets for now)

$$0 \to \mathcal{I}_S \to \mathcal{O}_{\mathbb{P}^n \times X} \to \mathcal{O}_S \to 0$$

tensor with $\pi^*_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}(m_P))$

$$0 \to \mathcal{I}_S(m_P) \to \mathcal{O}_{\mathbb{P}^n \times X}(m_P) \to \mathcal{O}_S(m_P) \to 0$$

push to X

$$0 \to \underline{\pi_{X,*}\mathcal{I}_S(m_P)} \to \pi_{X,*}\mathcal{O}_{\mathbb{P}^n \times X}(m_P) \to \pi_{X,*}\mathcal{O}_S(m_P) \to 0$$

Fibre of at $x \in X$: $H^0(\pi_X^{-1}(x), \mathcal{I}_S(m_P)|_{\pi_X^{-1}(x)})$

Scheme structure:

Write $H^0(m) := H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$.

Let $[V] \in Gr(\cdots)$ $\qquad V \subseteq H^0(m_p)$

There is a multiplication map

$$\times_k : V \otimes H^0(k) \longrightarrow H^0(k + m_p).$$

If $V = H^0(\mathbb{P}^n, \mathcal{I}_X(m_p))$ for some $X \subseteq \mathbb{P}^n$, then

$$\underbrace{\mathrm{im}(\times_k)}_{\dim \leq h_{\mathbb{P}^n}(m_p+k) - P(m_p+k)} \subseteq H^0(\mathbb{P}^n, \mathcal{I}_X(k+m_p))$$

$\quad \mathrm{rank}(\times_k) \leq h_{\mathbb{P}^n}(m_p+k) - P(m_p+k)$

Theorem: the equations $\mathrm{rank}(\times_k) \leq \overbrace{h_{\mathbb{P}^n}(m_p+k) - P(m_p+k)}^{N_k}$ for all k cut out $\mathcal{H}_{p,n}$ as a subscheme of the Grassmannian.

(Fact: always $\mathrm{rank}(\times_k) \geq N_k$.
Imposing the above for $k=1$ is enough)
↳ finitely many equations

Nice facts: $\mathcal{H}_{p,n}$ is connected and projective for any p.

Bad facts: Everything?

e.g. twisted cubic $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$
$C \qquad [s,t] \mapsto [s^3, s^2t, st^2, t^3]$

(not contained in any hyperplane)

$$k[x,y,z,w]/(xz-y^2, yw-z^2, xw-yz)$$

Hilbert poly. is $h_C(x) = 3x+1$

$[C] \in \mathcal{H}_{3x+1,3}$ (nice, but see next page)

- The double line $x=y=0$ squared ☹

$k[x,y,z,w]/(x^2,xy,y^2)$

(0)	(1)	(2)
dim 1	4	7

Hilbert poly is also $3x+1$

$[☹] \in \mathcal{H}_{3x+1,3}$

- X is a curve of degree d, genus g over $\mathbb{C}$

RR: $\chi(X, \mathcal{O}_X(n)) = \deg \mathcal{O}_X(n) + 1-g = dn+1-g$

$\mathcal{H}_{dn+1-g,\,r}$ = Hilbert scheme of degree d, genus g curves in $\mathbb{P}^r$

as a polynomial in n

but this still has things of genus $\neq g$

$(\mathcal{H}_{d,g,r})$

$\mathcal{M}_g = \mathcal{H}_{6(g-1),\,g,\,5g-6} /\!/ PGL(5g-5, \mathbb{C})$

this will give something for $g \geq 2$

For $\mathcal{M}_1$, one can construct things a bit differently.

$\mathcal{M}_1$: moduli space of curves of genus 1 (cubic curves)

parametrise a cubic curve by $\sum_{\substack{i+j+k=3 \\ i,j,k \geq 0}} c_{ijk} x^i y^j z^k$ $\quad c_{ijk} \in \mathbb{C}$

↳ has 10 coefficients

and they uniquely determine the curve g up to a scalar s

$\rightsquigarrow$ parameter space $\mathbb{P}(\mathbb{C}^{10}) = \mathbb{CP}^9$

More invariant way:

$$\mathbb{P}\left(\mathrm{Sym}^3(\mathbb{C}^3)^\vee\right) /\!/ \underbrace{SL(3,\mathbb{C})}_{\text{Linear equivalence}}$$

Step 1: Pick a 1-parameter subgroup $\lambda : \mathbb{C}^\times \to SL(3,\mathbb{C})$

Can diagonalise, i.e. pick abs x, y, z of $\mathbb{C}^3$ s.t.

$$\lambda(t) = \deg(t^a, t^b, t^c) \text{ for some } a+b+c=0.$$

$\mathrm{Sym}^3(\mathbb{C}^3)^\vee$ has a basis made out of $x^i y^j z^k$ for $i+j+k=3$, $i,j,k \geq 0$.

Pick $f \in \mathbb{P}(\mathrm{Sym}^3(\mathbb{C}^3)^\vee)$ $\quad f = \sum c_{ijk} x^i y^j z^k$

$$\lambda(t) f = \sum t^{ai+bj+ck} c_{ijk} x^i y^j z^k$$

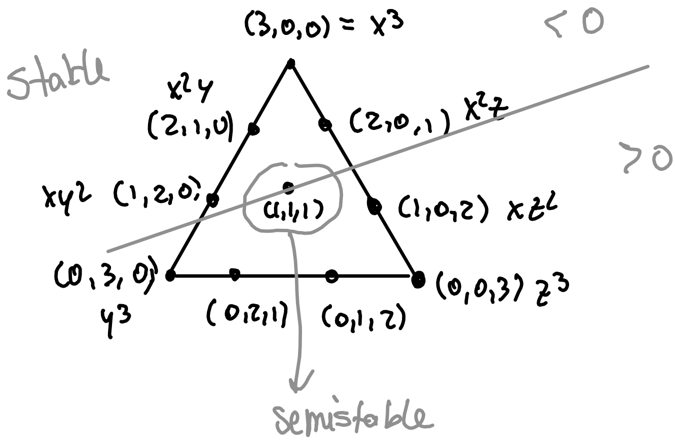
Step 2: I want to look at all terms $x^i y^j z^k$ inside of f where $\lambda(t)$ acts by something nonzero

$$\mu_\lambda(f) = \left\{ \begin{array}{l} \text{minimal value of } ai+bj+ck \\ \text{for some } c_{ijk} \neq 0 \end{array} \right\}$$

Sign determines μ-stability.

- f is λ-stable if $\mu_\lambda(f) < 0$
- f is λ-semistable if $\mu_\lambda(f) \leq 0$

Example: $(a,b,c) = (-5,1,4)$ $\qquad -5i + j + 4k = 0$



$$f = c_{300}x^3 + c_{210}x^2y + c_{201}x^2z + c_{120}xy^2 + c_{111}xyz +$$
$$+ c_{102}xz^2 + (y,z)$$

- $c_{300} = 0$, $f \in (y,z) \Rightarrow (1,0,0)$ on curve

...

$[f]$ stable iff f is smooth

$[f]$ semistable iff f has at most an ordinary double point

$[f]$ unstable iff cusp or worse