Stability

- stable points X^s
- semistable points X^{ss}
- unstable X^{us} (i.e. not semistable)

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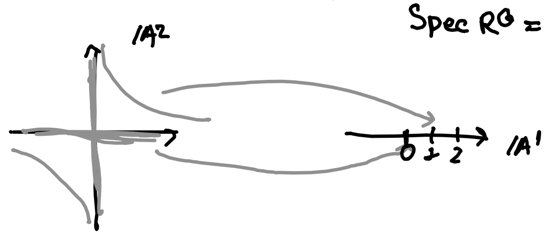
- Affine case: no unstable points, all are semistable

$x \in X^s$ if 1) $G \cdot x$ closed
2) G_x finite

<u>Ex:</u> $G = \mathbb{G}_m \curvearrowright \mathbb{A}^2$ $\quad t \cdot (x,y) \mapsto (tx, t^{-1}y)$ $\quad R^G = k[xy]$

$\mathrm{Spec}\, R^G = \mathbb{A}^1$

$\mathbb{A}^2 \longrightarrow \mathbb{A}^1$
$(x,y) \longmapsto xy$



$X^s = \{(x,y) \in \mathbb{A}^2 : xy \neq 0\}$ open subset

our GIT quotient restricted to X^s is a geometric quotient

- Projective case: X $\quad L$ (ample line bundle)

G-action $X \hookrightarrow \mathbb{P}^n$ is linear if

1. linear map $G \to GL_{n+1}$
2. a G-equivariant embedding $X \hookrightarrow \mathbb{P}^n$.

$X \hookrightarrow \mathbb{P}^n$ $\quad I(X)$

$\tilde{X} := \mathrm{Spec}\, R(X)$, $\quad R(X) = k[x_0, \dots, x_n]/I(X)$

$R(X)^G = \bigoplus_{r \geq 0} R(X)^G_r$, $\quad R(X)^G_+ = \bigoplus_{r > 0} R(X)^G_r$

N null cone: closed subscheme defined by $R(X)^G_+$

$X^{ss} = X \setminus N$

$x \in X^{ss}$, $\exists$ G-equivariant homogeneous polynomial $f \in R(X)^G_r$, $r>0$
s.t. $f(x) \neq 0$

X^{ss} is open

$$R(X)^G \hookrightarrow R(X)^{I(X)}$$

$$X \dashrightarrow \mathrm{Proj}(R(X)^G)$$

domain of this map is X^{ss}?

$X^{ss} \dashrightarrow \mathrm{Proj}\, R(X)^G$ GIT to the linear ...

$x \in X^s$, $\exists$ G-equivariant homogeneous polynomial $f \in R(X)^G_r$, $r>0$ s.t. $f(x) \neq 0$

2) G-action on X_f is closed (orbits are closed)
$\shortparallel$ $D(f)$

3) G_x finite

Criterion (easier to use):

$x \in X(k)$

$x \in X^s$ iff
1) $x \in X^{ss}$
2) $G \cdot x$ is closed in X^{ss}
3) G_x finite

satisfies ①, ② but not ③

$\hookrightarrow$ also poly-stable? did we define this? $X^{ps} := \{\text{poly-stable points}\}$

$\mathbb{G}_m \curvearrowright \mathbb{A}^2$ (0,0) is poly-stable

$x_1, x_2 \in X^{ss}$ ($\varphi : X \quad X/\!/G$)

$x_1 \underset{\text{S-equiv.}}{\sim} x_2$ iff $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \cap X^{ss} \neq \emptyset$
iff $\varphi(x_1) = \varphi(x_2)$

$X^{ss}/\sim_{\text{S-equiv.}}$ bijection on the sets $X/\!/G(k) \cong X^{ss}(k)/\sim_{\text{S-equiv.}} = X^{ps}(k)/G(k)$

Ex: $G = \mathbb{G}_m$, $X = \mathbb{P}^n$ $\quad t\cdot[x_0:\cdots:x_n] = [t^{-1}x_0 : tx_1 : \cdots : tx_n]$

$R(X) = k[x_0,\dots,x_n]$, $\quad R(X)^G = k[x_0x_1,\dots,x_0x_n]$
$\cong k[y_0,\dots,y_{n-1}]$

$\operatorname{Proj} R(X)^G = \mathbb{P}^{n-1}$

$R(X)^G_+ = (x_0x_1,\dots,x_0x_n)$

$N = \{[x_0:\cdots:x_n] \in \mathbb{P}^n : x_0 = 0 \text{ or } x_1 = \cdots = x_n = 0\}$ $\qquad \cong \mathbb{A}^n \setminus 0$?

$X^s = X^{ss} = X \setminus N = \bigcup_{i=1}^{n} X_{x_0x_i} = \mathbb{A}^n \setminus \{0\}$

• <u>what if $X \cancel{\hookrightarrow} \mathbb{P}^n$ not automatically?</u>

X projective scheme, L ample line bundle
sections of $L^{\otimes m}$, $m \geq 0$ give us an embedding

$$X \hookrightarrow \mathbb{P}^n = \mathbb{P}(H^0(X, L^{\otimes m})^*)$$

also want: the embedding to be G-equivariant!

The <u>linearisation</u> wrt $\sigma : G\times X \to X$ is a line bundle
$\pi : L \to X$ with $G\times L \cong \sigma^* L$.

$\pi_X^* L$

$\pi_X : G\times X \to X$ projection

$\tilde{\sigma} : G\times L \to L$ is

$$\begin{tikzcd}
G\times L \arrow[r, "\cong"] \arrow[rr, bend left, "\tilde{\sigma}"] \arrow[dr, bend right, "\mathrm{id}_G\times\pi"'] & \sigma^* L \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L \arrow[d, "\pi"] \\
& G\times X \arrow[r, "\sigma"'] & X
\end{tikzcd}$$

These are true:

1. $\pi : L \to X$ is G-equivariant
2. $\forall g \in G,\ x \in X \quad L_x \longrightarrow L_{g \cdot x}$ is linear
3. $\tilde{\sigma} : G \times L \to L$ induces a linear representation
   $G \longrightarrow GL(H^0(X, L))$.

$R(X,L) = \bigoplus_{r \geq 0} H^0(X, L^{\otimes r})$ (here L is ample)

$R(X,L)^G = \bigoplus_{r \geq 0} H^0(X, L^{\otimes r})^G$

$\mathrm{Proj}\left(R(X,L)^G\right)$

$X^{ss}(L) = \{x \in X : \exists\, \alpha \in H^0(X, L^{\otimes r})^G \text{ for some } r > 0,\ \alpha(x) \neq 0\}$

f ~~G-equiv.~~ homogeneous

$x \in X^s(L)$ iff $\begin{cases} x \in X^{ss}(L) \\ G_x \text{ finite} \\ \text{the } G\text{-orbits on } X_\alpha \text{ are closed} \end{cases}$

$X^{ss}(L) \longrightarrow \mathrm{Proj}\, R(X,L)^G \qquad X /\!\!/_L G$

$x_1, x_2 \in X^{ss} \quad x_1 \underset{S\text{-equiv}}{\sim} x_2 \qquad \overline{G \cdot x_1} \cap \overline{G \cdot x_2} \cap X^{ss} \neq \emptyset$

↳ similar results as before