Algebraic groups and affine GIT

[Campbell Brawley]

Goal: $G \subseteq GL_n(k)$, $G \circlearrowright X \leftarrow$ variety

build a suitable quotient X/G

1. Algebraic groups
2. Actions and representations
3. Quotients
4. Reductive groups

(1) What is an algebraic group?

An algebraic group looks like:

$GL_n(k)$, $SL_n(k)$, $SU_n(\mathbb{C})$, ...

Definitions:

Option 1 An algebraic group is a group object in the category of k-Schemes.

$$\underbrace{G,\quad m: G\times G \longrightarrow G}_{\text{multiplication}},\quad \underbrace{i: G \longrightarrow G}_{\text{inversion}},\quad \underbrace{e: \operatorname{Spec} k \longrightarrow G}_{\text{identity}}$$

Option 2 An algebraic group is a variety over k which is also a group such that $m(x,y) = xy$ $\quad i(x) = x^{-1}$

are maps of varieties.

We will only care about G affine.

Examples:

- $GL_n(k)$ is an affine variety described

$$GL_n(k) = Z(\det(x_{ij})\, y - 1) \subset \mathbb{A}_k^{n^2+1}$$

- The other examples above follow by adding polynomials.

  e.g. $SL_n(k)$ $y-1$

- $GL_n(k) = \mathbb{G}_m(k) \simeq k^\times$, $\mathbb{G}_a(k) \simeq (k,+)$

Note: Associated to the affine scheme G is its coordinate ring $k[G]$. By the equivalence between k-algebras and (affine) k-schemes, the group structure on G gives us some structure on k

$$\begin{cases} m: G\times G \longrightarrow G \\ i: G \longrightarrow G \\ e: \operatorname{Spec} k \to G \end{cases} \rightsquigarrow \begin{cases} k[G] \to k[G]\otimes k[G] \\ k[G] \to k[G] \\ k[G] \to k \end{cases}$$

This makes $k[G]$ into a Hopf algebra.

(2) Actions of algebraic groups

Definition: X k-scheme, G

G acts on X if it acts on X in the sense of group theory such that the action map $G\times X \longrightarrow X$, $(g,x)\mapsto g\cdot x$ is a map of k-schemes.

We call X a G-space or G-scheme.

<u>Orbits</u> and <u>stabilisers</u> are defined in the obvious way.

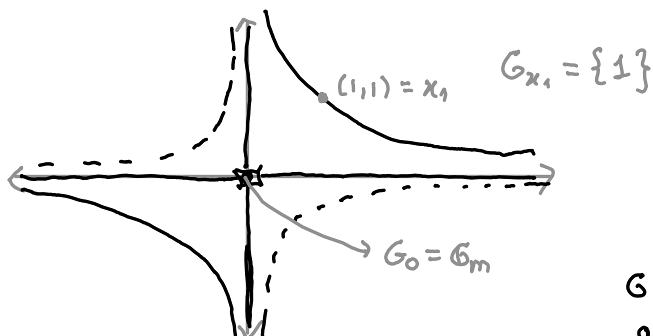
Also: $G \circlearrowleft X \longrightarrow G \circlearrowleft \mathcal{O}_X(X)$

by $(g\cdot f)(x) = f(g^{-1}x)$.

The <u>(rational) representation</u> of G is a map of algebraic groups

$G \longrightarrow GL(V)$ for V a finite-dimensional vector space.

<u>Examples:</u> $\underset{(k^\times)}{\mathbb{G}_m(k)} \circlearrowleft \underset{(k^2)}{\mathbb{A}^2_k}$ by $a\cdot(x,y) = (ax, a^{-1}y)$



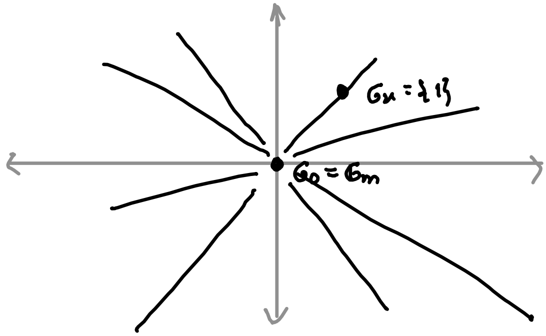
$G \longrightarrow X$

$g \longmapsto g\cdot x$

<u>observation:</u>

- Not all of the orbits are closed!
- $\boxed{\dim G_x + \dim G\cdot x = \dim G}$ Fact

$\mathbb{G}_m(k) \circlearrowright \mathbb{A}^2_k$, $\quad a\cdot(x,y) = (ax, ay)$



<u>General facts:</u>

- $\overline{G\cdot x} \setminus G\cdot x$ is a union of orbits of strictly smaller dimension.
- Each $\overline{G\cdot x}$ contains a closed orbit of minimal dimension.

③ <u>Quotients</u>

Affine $G \circlearrowright X$ scheme

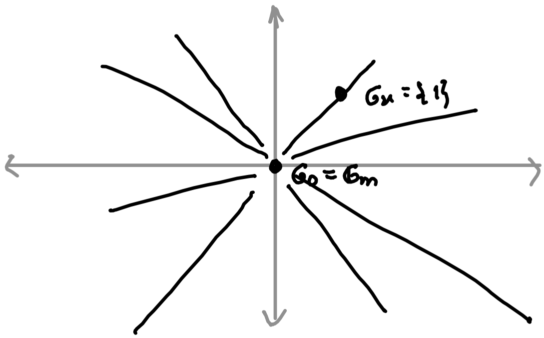
We want to define a quotient X/G.

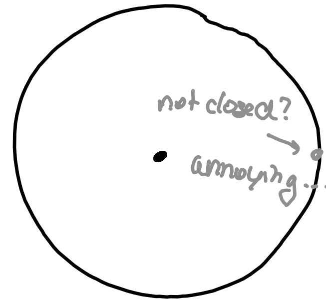
<u>Definition:</u> $\varphi : X \to Y$, G-invariant $(\varphi(g\cdot x) = \varphi(x))$ is a <u>categorical quotient</u> if for any G-invariant $f: X \to Z$,

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "f"'] & Y \arrow[d, dashed, "\exists!"] \\ & Z \end{tikzcd}$$

<u>Idea:</u> The set of orbits!

Example: $\mathbb{G}_m \circlearrowright \mathbb{A}^2_k$, $a\cdot(x,y) = (ax, ay)$





In this case, there is no variety structure on X/G such that $X \longrightarrow X/G$ is a map of varieties.

We learn: Taking the orbit space is problematic.

Def: X G-space

A good quotient is a G-invariant map $\varphi : X \longrightarrow Y$ satisfying

1) φ surjective

2) $\mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\varphi^{-1}(U))$ maps isomorphically onto $\mathcal{O}_X(\varphi^{-1}(U))^G$
for all $U \subseteq Y$ open

action defined on germs

3) $W \subseteq X$ G-invariant closed $\rightsquigarrow$ $\varphi(W)$ closed

4) $W_1, W_2 \subseteq X$ disjoint closed G-invariant
$\rightsquigarrow$ $\varphi(W_1)$, $\varphi(W_2)$ disjoint

5) φ affine.

Theorem: A good quotient is a categorical quotient.

How do we construct a good quotient? X affine

The hint comes from 2) above. $\mathcal{O}_Y(Y) \cong \mathcal{O}_X(X)^G$

Definition: Let X be an affine G-space.
The affine GIT quotient is $X /\!/ G := \operatorname{Spec}(\mathcal{O}_X(X)^G)$.

It is not difficult to see that this is a categorical quotient.

However, not much can be said about $X/\!/G$.

We might hope that if $\mathcal{O}_X(X)$ is a fin. gen. k-algebra then the same is true of $\mathcal{O}_X(X)^G$.

↳ This is not the case in general!

We need to restrict to reductive groups.

4 Reductive groups

Key point: Reductive groups are ones for which we can prove niceness properties about their GIT quotients.

Recall: A matrix $M \in GL_n(k)$ is semisimple if it is diagonalisable (i.e. it acts semisimply on k^n), unipotent if it is conjugate to some matrix $\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

Every $M \in GL_n(k)$ can be written as $M = M_{ss} M_u$ with M_{ss} semisimple, M_u unipotent, M_{ss} and M_u commuting.

Theorem: (Jordan decomposition)

G affine algebraic group

Then every $g \in G$ may be written $g = g_{ss} g_u = g_u g_{ss}$, where $\varphi(g_{ss})$ (resp. $\varphi(g_u)$) is semisimple (resp. unipotent) for any representation φ of G.

Definition: An affine algebraic group is <u>reductive</u> if it is smooth and all smooth unipotent normal subgroups are trivial.

Theorem: If $\mathrm{char}(k) = 0$, then G reductive $\Leftrightarrow$ every representation of G is semisimple.

Theorem: (Nagata, Hilbert)

If G is a reductive algebraic group over k acting on a finitely-generated k-algebra A, then A^G is finitely generated.

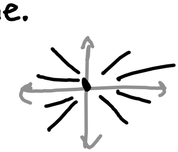
$$G \circlearrowright X \qquad X/\!/G$$

Theorem: G reductive, X affine

$G \circlearrowright X \Rightarrow X/\!/G$ is a good quotient and is affine.

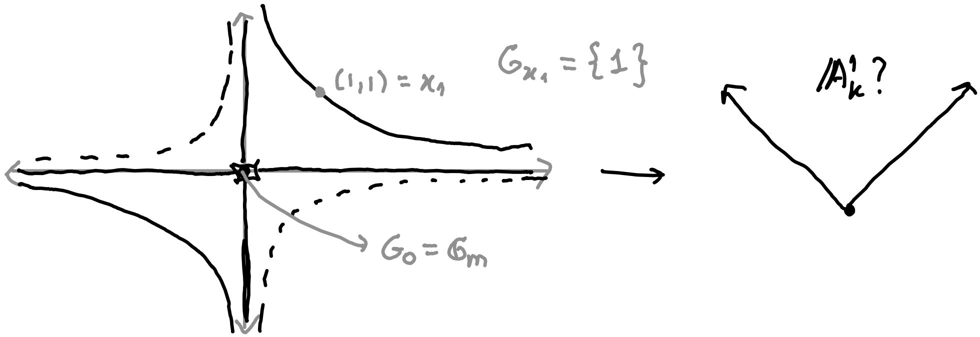
Facts: • $f(x) = f(x') \Leftrightarrow \overline{G\cdot x} \cap \overline{G\cdot x'} \neq \emptyset$

(not $\overline{G\cdot x} = \overline{G\cdot x'}$)



• set theoretically, $X/\!/G$ is the set of closed orbits of X.

<u>Examples:</u> $\mathbb{G}_m(k) \circlearrowright \mathbb{A}^2_k \qquad a(x,y) = (ax, a^{-1}y)$



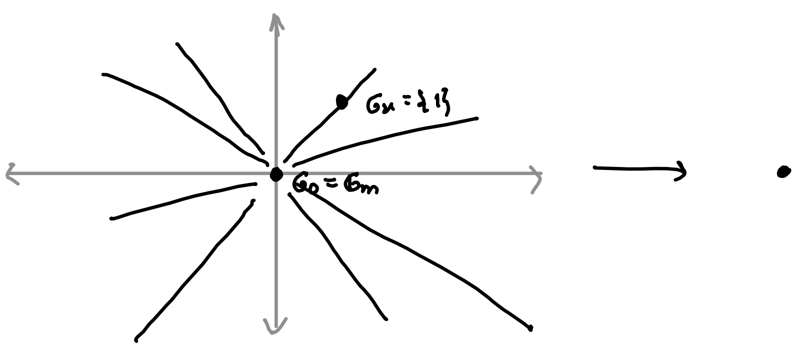
Algebraically: $k[x,y]^G$ $\quad x \mapsto ax$, $y \mapsto a^{-1}y$

$\cong$

$k[xy]$

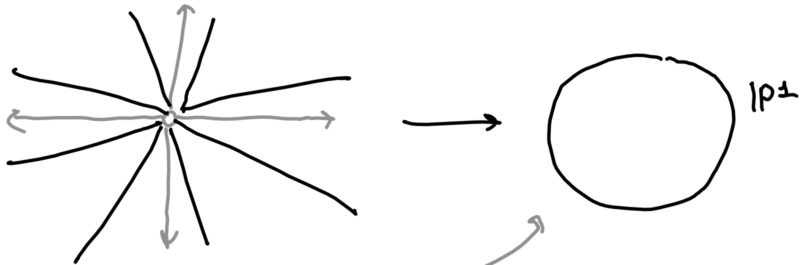
so indeed we get $\mathbb{A}^2_k /\!/ G \cong \mathbb{A}^1_k$.

$\mathbb{G}_m(k) \circlearrowright \mathbb{A}^2_k \qquad a(x,y) = (ax, ay)$



$k[x,y]^G \cong k$, so indeed we get $\mathbb{A}^2_k /\!/ \mathbb{G}_m \cong \operatorname{Spec} k$.

$\mathbb{G}_m(\mathbb{C}) \circlearrowright \mathbb{A}^2_{\mathbb{C}} \setminus \{0\}$ $\quad a \cdot (x,y) = (ax, ay)$ $\quad$ non-affine example



$\mathbb{C}[x,y]^G_{(x,y)} \simeq \mathbb{C}$ global sections

In the non-affine case, this construction doesn't work.

Come to the next talk to find out what to do here :)