Throughout, we work over $\mathbb{C}$.

Prop. Let G be a complex Lie group. Then G is reductive iff $G = K^{\mathbb{C}}$ for a maximal compact subgroup $K \leq G$.

Here, $G = K^{\mathbb{C}}$ means $\mathrm{Lie}(G) = \mathrm{Lie}(K) \otimes_{\mathbb{R}} \mathbb{C}$.

Example

K	G
S^1	$\mathbb{C}^*$
$SU(n)$	$SL(n, \mathbb{C})$
$U(n)$	$GL(n, \mathbb{C})$
$\vdots$	

In the last talk, we had:

Thm Suppose $K \curvearrowright (X, \omega)$ with moment map μ. Suppose $K \curvearrowright \mu^{-1}(0)$ freely. Then $\mu^{-1}(0)/K$ is a symplectic manifold.

Remark What happens if the action is not free?

Then we get a stratified symplectic space.

i.e. $$\mu^{-1}(0)/K = \coprod_{\alpha \in A} U_\alpha$$

where: each U_α is a symplectic mfd

$$\overline{U_\alpha} = \bigcup_{\beta \leq \alpha} U_\beta .$$

(A is a poset).

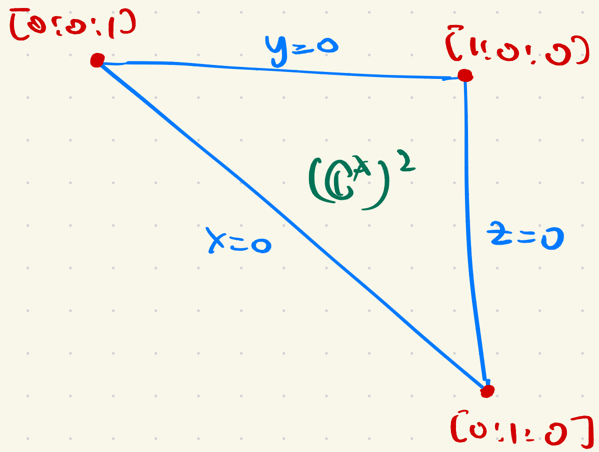
<u>Example</u> Say $\mathbb{P}^2$ with coords $[x:y:z]$

$T^2 \curvearrowright \mathbb{P}^2$ by

$$(s,t)[x:y:z] = [x:sy:tz].$$

Then we get

$\mathbb{P}^2 =$



Back to symplectic reduction:

$$\dim_{\mathbb{R}}(\mu^{-1}(0)/K) = \dim_{\mathbb{R}}(X) - 2\dim_{\mathbb{R}}(K)$$
$$= \dim_{\mathbb{R}}(X) - \dim_{\mathbb{R}}(G).$$
$$= \text{"}\dim_{\mathbb{R}}(\underbrace{X/G}_{})\text{"}.$$

what we want is the GIT quotient...

<u>Idea</u> we have a natural map

$$\mu^{-1}(0)/K \longrightarrow X/G.$$

what is this map?

Back to GIT.

Let $L \to X$ be an ample line bundle.

Then we have a GIT quotient

$$X/\!\!/_L G = \underbrace{X^{L\text{-}ss}/\!\!/G}_{\text{alg.}} = X^{L\text{-}ps}/G$$

$X^{L\text{-}ss} = \{ x \in X \mid x \text{ is semistable wrt linearisation } L \}.$

resp. $X^{L\text{-}ps}$

Kempf-Ness

What we want to show is:

1. $\mu^{-1}(0) \subseteq X^{L\text{-}ss}$

   So we get a natural map

$$\mu^{-1}(0)/K \longrightarrow \underbrace{\left(X^{L\text{-}ss} /\!/_{alg} G\right)_{an.}}_{=(X/\!/_L G)_{an.}}$$

2. The above map is an isomorphism, ie

   it sends strata on LHS to orbit on RHS,

   making it Kähler.

Remark This is hopeless in this generality.

L and μ need to be related for this to make sense.

In nice cases, 1. was proven by Kempf-Ness.

2. was proven by Kirwan.

Analytic stability

Define

$$X^{\mu\text{-ss}} = \{ p \in X \mid \overline{G \cdot p} \cap \mu^{-1}(0) \neq \emptyset \} \subseteq X.$$

For the analytically semistable orbits.

<u>Thm</u> There is a categorical quotient of complex analytic spaces $X^{\mu\text{-ss}} /\!\!/_{an} G$.

Moreover, $\mu^{-1}(0)/K \longrightarrow X^{\mu\text{-ss}} /\!\!/_{an} G$ is a homeo.

This proves 2.

In terms of 1, note that

$$\left(X^{L\text{-ss}} /\!\!/_{alg.} G \right)_{an} = X^{L\text{-ss}} /\!\!/_{an} G$$

and so it would suffice to show

$$X^{\mu\text{-ss}} = X^{L\text{-ss}}.$$

ie analytic stability = algebraic stability.

This is the Kempf-Ness theorem.

For concreteness, suppose we had:

$$\begin{array}{ccccc} K & \subseteq & G & \circlearrowleft & X \\ \cap\!\! | & & \cap\!\! | & & \cap\!\! | \\ SU(N+1) & \subseteq & SL(N+1) & \circlearrowleft & \mathbb{P}^N \end{array}$$

and we take $L = \mathcal{O}(1)$, $\omega = \omega_{FS}|_X$.

In this case, we have a canonical moment map

$$\mu_{FS} : \mathbb{P}^N \longrightarrow \mathfrak{su}(N+1)^* \cong \mathfrak{su}(N+1)$$

$$\mu_{FS}(x) = \frac{i(\langle \cdot, \tilde{x}\rangle \otimes \tilde{x})_0}{\|\tilde{x}\|^2}.$$

$(\cdot)_0$ = trace free part.

We then obtain a moment map μ on X via

$$X \hookrightarrow \mathbb{P}^N \xrightarrow{\mu_{FS}} \mathfrak{su}(N+1)^* \longrightarrow \mathfrak{k}^*.$$

<u>Thm</u> (Kempf-Ness). Let $x \in X$. Then

(i) $x \in X^{L\text{-}ss} \iff \overline{G \cdot x}$ contains a 0 of μ

(ii) $x \in X^{L\text{-}ps} \iff G \cdot x$ contains a 0 of μ.

If so, then $G \cdot x \cap \mu^{-1}(0)$ is a single K-orbit.

Pf Consider the function

$$\mathcal{M}_x : G/K \longrightarrow \mathbb{R}$$
$$[g] \longmapsto \log|g.\hat{x}|$$

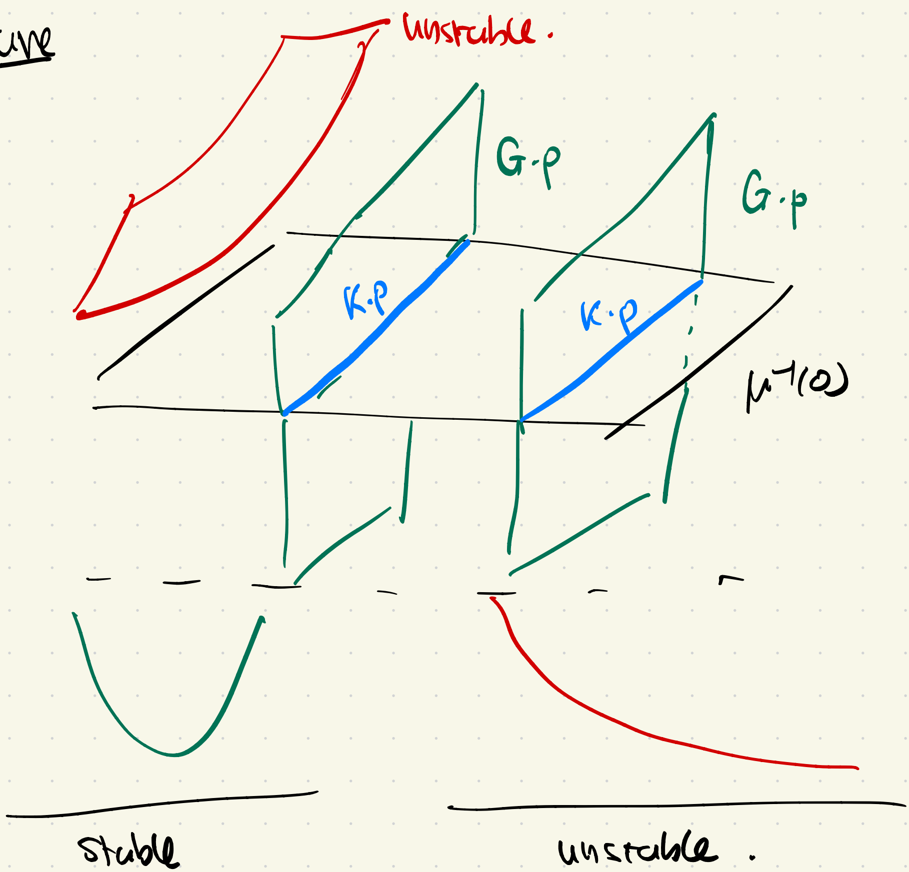
well defined as $K \subseteq SU(N+1)$.

G/K is a non-pos. curved symm. space.

Then: (i) $[g]$ is a critical pt of $\mathcal{M}$ iff $\mu(g.x)=0$

(ii) $\mathcal{M}$ is convex along geodesics in G/K. $\square$

Picture



<u>Conclusion</u> We have three ways to check stability.

- Topological
- Numerical (Hilbert–Mumford)
- Analytic (Kempf-Ness, Log-norm functional, ...)

These agree in our examples.

For "infinite dimensional GIT," they don't necessarily agree.

<u>Aside</u> A hyperkähler mfd X consists of a metric g,

ACS I, J, K s.t. $I^2 = J^2 = K^2 = IJK = -\mathrm{id}$.

such that (X, g, I), (X, g, J), (X, g, K) are kähler.

Suppose H is a compact Lie group, $H \curvearrowright X$ preserving everything, with moment maps

$$\mu_I, \mu_J, \mu_K.$$

then we get the hyperkähler quotient

$$\frac{\mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)}{H}.$$

Now if we set

$$\mu_{\mathbb{C}} = \mu_J + i\mu_K$$

This is holomorphic wrt I.

Thus, $\mu_{\mathbb{C}}^{-1}(0)$ is a complex submanifold of X.

$H \circlearrowright \mu_{\mathbb{C}}^{-1}(0)$, with moment map $\mu_I|_{\mu_{\mathbb{C}}^{-1}(0)}$

$\Rightarrow$

$$\frac{\mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)}{H} = \frac{\mu_I^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0)}{H}$$

$$\text{(Kempf–Ness)} = \frac{\mu_{\mathbb{C}}^{-1}(0)}{H^{\mathbb{C}}}$$

Morally, "hyperkähler quotient = complex symplectic reduction".

Infinite dimensional GIT

In this case, K makes sense, but often G does not exist. However, the stability conditions still make sense.

In most settings, $\mu^{-1}(0)$ is a PDE, and so what Kempf-Ness says is

"Solution to PDE exists $\iff$ some stability condition holds".

<u>Example</u> Let $E \to \Sigma$ be a hol. v.b., Σ a R.s.

[Narasimhan-Seshadri, Donaldson]

Suppose $\deg E = 0$. Then

E is stable $\iff$ E admits a flat connection.

More generally, Hermitian-Yang-Mills connections:

[Atiyah-Bott]

HYM is a moment map,

$$\frac{\{\text{connections}\}}{G} \cong \frac{\{\text{minimum of YM-functional}\}}{K}.$$

[Hitchin-Kobayashi conjecture, proven by Donaldson-Uhlenbeck-Yau]

E is slope polystable $\iff$ E admits a HYM connection.

(Σ can be any compact Kähler manifold).

Example (Yau-Tian-Donaldson Conjecture)

K-polystability $\iff$ constant scalar curvature metrics.