Geometric invariant theory

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<u>Setup</u>: $G \curvearrowright X$, what is X/G?

Why are we interested? Moduli spaces!

1. $\mathcal{M}_g \sim$ smooth projective algebraic curves (genus g)

   $\dim = 3g-3$

2. $\mathrm{VecBund}_X^{r,d} \sim$ vector bundles over X — you can fix rank, degree (if you want)

3. $\mathrm{PrincBund}_X^G$

What is a (fine/coarse) moduli space?

Space $\mathcal{M}$ in bijection with the objects we want (their isomorphism classes)

$\mathcal{M}$ should "respect" how objects "vary" in "families"

Curves:

$$\begin{array}{c} \mathcal{C} \\ \downarrow \pi \\ S \end{array}$$

<u>Flat</u>

$\pi^{-1}(s) \cong$ smooth projective algebraic curve of genus g

Bundles: (over X)

$$\begin{array}{c} \mathcal{E} \\ \downarrow \\ X \times S \end{array}$$

$s \in S$

$\mathcal{E}|_{X \times s}$ is v.b. over X

Fine moduli space $\mathcal{M}$: (variety/scheme)

$\exists$ family $\begin{matrix}\mathcal{U}\\ \downarrow \\ \mathcal{M}\end{matrix}$ such that $\forall$ families $\begin{matrix}\mathcal{C}\\ \downarrow \\ \mathcal{S}\end{matrix}$ $\exists !\ \mathcal{S} \to \mathcal{M}$ s.t.

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{U} \arrow[d] \\ \mathcal{S} \arrow[r] & \mathcal{M} \end{tikzcd}$$

Examples: similar oriented triangles

stable vector bundles, principal bundles

Grassmannians $Gr_3(V)$, $Gr_1(V) = \mathbb{P}(V)$

Problem: $\begin{matrix}\mathcal{C}\\ \downarrow \pi \\ \end{matrix}$ "constant" family

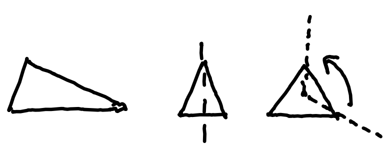
If $\mathcal{C}_s = \pi^{-1}(s)$ has a non-trivial automorphism: then $\mathcal{C}_s \overset{\varphi}{\cong} \mathcal{C}_s$

$$\begin{matrix}\mathcal{C}'\\ \downarrow \pi\end{matrix}$$

glue using φ

contradiction!

e.g. similar triangles



How to fix this?

1. Wish for less (coarse moduli space)
2. Incorporate more data into objects, killing automorphisms

GIT can help!

3. Algebraic stacks

Coarse moduli space $\mathcal{M}$

(*) $\forall$ families $\begin{array}{c}\mathcal{C}\\ \downarrow \\ S\end{array}$, induce $S \to \mathcal{M}$ (not necessarily unique!)

- $\mathcal{M}$ has the "finest" topology s.t. (*) is true
- $\forall \mathcal{M}'$ fulfills (*), $\exists\, \mathcal{M} \longrightarrow \mathcal{M}'$

(here we don't have to restrict to stable)

Back to GIT

Wishlist

$G \curvearrowright X$. A quotient "$f : X \longrightarrow X/G$"

1. X/G is a variety (at worst maybe a scheme)
2. $X/G = \{ Gx \mid x \in X \}$ (geometric quotient)
3. f is G-invariant, $(f_* \mathcal{O}_X)^G = \mathcal{O}_{X/G}$
   (good quotient)
4. $f : X \to X/G$ is categorical quotient

   $\forall h : X \to Y$, G-inv.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Y \\ X/G \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

1, 3 $\Rightarrow$ 4 (Hoskins Prop 3.30)

$$X \supset X^{ss} \supset X^{s}$$

X^{ss}: semistable, X^{s}: stable

Conditions

- G is reductive (linear algebraic group)

  e.g. GL, SL, SO, Sp $\qquad\qquad G \curvearrowright X$

Defi: (a) $x \in X$ is <u>GIT-stable</u> if $\dim(G_x) = 0$ (stabiliser)

($G \cdot x$ is of maximal dimension)

(b) In projective GIT, $X^{ss} = X \setminus N$

N = vanishing locus of $(K[X]_+)^G$ in X

[(a) $\Rightarrow$ (b)]

Affine
$X \to X/G$
Projective
$X \dashrightarrow X/G$

We also have "polystable", somewhere in between (a) and (b).