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1. Derived Functors

- $F: \mathcal{A} \to \mathcal{B}$ is right-exact $(L_nF)(A)$ by choosing a proj. res. $P \to A$

  & then define $L_nF(A) := H_n(F(P))$

  - This is well-defined
  - This is a functor. It's additive.

  } Comparison Theorem.

- L_nF commute w/ exact functors. $\boxed{U \text{ exact}}$

$$(L_nUF)(A) \cong U(L_nFA)$$

$$0 \to \ker(f) \to A \xrightarrow{f} B \rightsquigarrow 0 \to U(\ker(f)) \to UA \xrightarrow{Uf} UB$$

$$U(\ker(f)) \cong \ker(Uf)$$

$$\operatorname{im}(A \xrightarrow{f} B) = \ker(\operatorname{coker}(A \xrightarrow{f} B))$$

$$(L_nF)(A): \quad \dots \to P_1 \to P_0 \to A \to 0 \rightsquigarrow \dots \to FP_1 \to FP_0 \to 0$$

$$\downarrow$$

$$\dots \to UFP_1 \to UFP_0 \to 0$$

- $(L_nF)_{n\in\mathbb{N}}$ form a collection of δ-functors

$$0 \to A' \to A \to A'' \to 0$$

$$\begin{tikzcd} & L_0F(A') \arrow[d, phantom, "\cong"] & \\ & F(A') \arrow[r] & F(A) \arrow[r] & F(A'') \arrow[d] \\ & & & 0 \end{tikzcd}$$

$$\begin{tikzcd} \cdots \arrow[r] & L_iF(A') \arrow[r] & L_iF(A) \arrow[r] & L_iF(A'') \arrow[r, "\partial"] & L_{i-1}F(A') \arrow[r] & \cdots \end{tikzcd}$$

Sketch:

$$\begin{tikzcd} 0 \arrow[r] & P' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & P'' \arrow[r] \arrow[d] \arrow[l, bend right, "\sigma"'] & 0 \\ 0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A'' \arrow[r] & 0 \end{tikzcd}$$

$$0 \to P'_n \to P_n \twoheadrightarrow P''_n \to 0$$

projective

$$\begin{tikzcd} \text{so } 0 \arrow[r] & F(P') \arrow[r] & F(P) \arrow[r] & F(P'') \arrow[r] \arrow[l, bend right, "F\sigma"'] & 0 \end{tikzcd}$$

then apply LES in homology to this.

- $(L_nF)_{n\in\mathbb{N}}$ are in fact <u>universal</u> δ-functors

$$\varphi_0 : T_0 \to F \rightsquigarrow (\varphi_n : T_n \to L_nF)$$

Sketch: $\varphi_n : T_n \to L_n F$ natural transf.

comprises $T_n(A) \to L_n F(A)$ for each $A \in \mathcal{A}$

Pick $A \in \mathcal{A}$. Suppose φ_i for $0 \leq i < n$ have been constructed.

Choose $0 \to K \to P \to A \to 0$

projective

Look at:

$$\begin{tikzcd}
 & T_n(A) \arrow[r, "\partial_n"] \arrow[d, "\varphi_n"'] & T_{n-1}(K) \arrow[r] \arrow[d, "\varphi_{n-1}"] & T_{n-1}(P) \arrow[d, "\varphi_{n-1}"] \\
0 \arrow[r] & L_n F(A) \arrow[r, "\partial_n"] & L_{n-1} F(K) \arrow[r] & L_{n-1} F(P) \\
L_n F(P) \arrow[u, equal] & & &
\end{tikzcd}$$

$a \mapsto \partial_n a \mapsto 0$

$\varphi_n(a) \mapsto \underline{\varphi_{n-1} \partial_n a} \mapsto 0$

$\square$

2. Adjoints

- $L : \mathcal{C} \to \mathcal{D}$, $R : \mathcal{D} \to \mathcal{C}$

$L \dashv R$ "L is l. adjoint to R"

if $\forall A \in ob(\mathcal{C})$, $B \in ob\,\mathcal{D}$ we have

$$\mathrm{Hom}_{\mathcal{D}}(L(A), B) \cong \mathrm{Hom}_{\mathcal{C}}(A, R(B))$$

$\mathcal{C}^{op} \times \mathcal{D} \to \mathrm{Set}$ $\qquad$ $\mathcal{C}^{op} \times \mathcal{D} \to \mathrm{Set}$

$f : A \to A'$ in $\mathcal{C}$, $g : B \to B'$ in $\mathcal{D}$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(L(A'), B) \arrow[r, "\cong"] \arrow[d, "(Lf)^* \circ g_*"] & \mathrm{Hom}_{\mathcal{C}}(A', R(B)) \arrow[d, "f^* \circ (Rg)_*"] \\
\mathrm{Hom}_{\mathcal{D}}(L(A), B') \arrow[r, "\cong"] & \mathrm{Hom}_{\mathcal{C}}(A, R(B'))
\end{tikzcd}$$

Description 1 of Adjoints.

<u>Description 2</u>: $\mathrm{Hom}(LC, LC) \cong \mathrm{Hom}(C, RLC)$

$$1_{LC} \longmapsto \eta_C : C \to RLC$$

& similarly you can obtain

$$\varepsilon : LRC \to C.$$

$$\mathrm{Hom}(LC(C), D) \cong \mathrm{Hom}(C, R(D))$$

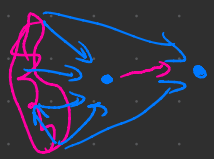
$$L(C) \xrightarrow{f} D \longmapsto C \xrightarrow{\eta_C} RL(C) \xrightarrow{Rf} RD$$

We should view $\left.\begin{array}{l}\eta : Id_{\mathcal{C}} \to RL, \\ \varepsilon : LR \Rightarrow Id_{\mathcal{D}}\end{array}\right\}$ nat. transforms

s.t.

$$\begin{tikzcd} R \arrow[r, Rightarrow, "\eta_R"] \arrow[dr, Rightarrow, "id"'] & RLR \arrow[d, Rightarrow, "R\varepsilon"] \\ & R \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, Rightarrow, "L\eta"] \arrow[dr, Rightarrow, "id"'] & LRL \arrow[d, Rightarrow, "\varepsilon_L"] \\ & L \end{tikzcd}$$

- Limits & Colimits are a thing.

$I \xrightarrow{F}$ 

$\mathrm{colim} : [I, \mathcal{C}] \to \mathcal{C}$

$\Delta : \mathcal{C} \to [I, \mathcal{C}]$ (ΔA sends all of I to A)

$\mathrm{colim} \dashv \Delta$ ← The defining property of colim.

- "Left adjoints preserve colimits":

$$[L(\mathrm{colim}\, F), B] \cong [\mathrm{colim}\, F, RB]$$
$$\cong [F, \Delta(RB)]$$
$$\cong [F, R \circ \Delta B]$$
$$\cong [LF, \Delta B] \cong [\mathrm{colim}\,(LF), B]$$

then apply Yoneda to deduce that

$$L(\mathrm{colim}\, F) \cong \mathrm{colim}\,(LF).$$

$\square$

- <u>Classic example</u>: $(-) \otimes_R A \dashv \mathrm{Hom}_R(A, -)$

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$$

(Eilenberg-Watts) $F : R\text{-Mod} \to S\text{-Mod}$ s.t. it's additive & preserves colimits then in fact

$$F \cong (-) \otimes_R B \text{ for } B \text{ an } (R, S)\text{-bimodule}$$

<u>Sketch</u>: Main idea is to construct $\sigma_M : M \times FR \to FM$ for each M. First, $m \in M$ & $\mu_m(r) = rm$.

$$R \to M$$

$$\sigma_M : M \times FR \to FM \rightsquigarrow F\mu_m : FR \to FM$$

This is bilinear & so yields $\sigma_M : M \otimes_R FR \to FM$

map of S-modules

Set $M = R$, you get FR having an (R,S)-bimodule structure.

<u>Key claim</u>: σ_M isomorphism.

↳ σ_R isom.

↳ apply some 5-lemma magic. $\square$

3. Examples

1. $\mathbb{Z}[-] : Ab \to Sh(X)$ "constant sheaf functor"

$\Gamma : Sh(X) \to Ab$ "global sections functors"

$\boxed{\mathbb{Z}[-] \dashv \Gamma}$ so... Γ is left exact!

$$H^i(X, \mathcal{F}) := (R^i \Gamma)(\mathcal{F}).$$

Theoretical aside: (F right exact)

An object Q is F-acyclic if $(L_i F)(Q) = 0$ for all $i > 0$.

If you have an SES

$$0 \to M \to \underset{\uparrow}{Q} \to A \to 0$$

F-acyclic

By LES, $(L_i F)(A) \cong (L_{i-1} F)(M)$ for $i \geq 2$.

Then if $\cdots \to Q_1 \to Q_0 \to A \to 0$

is a F-acyclic resolution of A then by considering

$$\left.\begin{array}{l} 0 \to M_m \to Q_m \to \cdots \to Q_0 \to A \to 0 \\ \qquad\qquad \cdots \to Q_{m+1} \to M_m \to 0 \end{array}\right\}$$

you obtain that $(L_{m+1} F)(A) \cong H_{m+1}(F(Q))$.

Weibel claims: A abelian group, $x_*(A)$ is I-acyclic

$x \in X$ is a point, this is the skyscraper sheaf.

$$0 \to A \to I_0 \to I_1 \to \cdots \quad \text{(inj. resolution)}$$

then $0 \to x_*(A) \to x_*(\underset{\uparrow}{I_0}) \to \cdots$ (inj. res.)

inj. object in $Sh(X)$

Computing $(R^i\Gamma)(x_*A)$ then looks at the original res. in Ab, so Γ-acyclic.

(see flasque/ sheaves)
flabby

2. $\mathrm{Ext}^i_R(A,B) := [R^i \mathrm{Hom}(A,-)](B)$

- A proj.
- $\mathrm{Hom}(A,-)$ exact
- $\mathrm{Ext}^i_R(A,B)=0$ for $i>0$ & any B.
- $\mathrm{Ext}^1_R(A,B)=0$ any B.

key point: induct using $\mathrm{Hom}(A,-)$ being left-exact already.

Similarly:
- A inj.
- $\mathrm{Hom}(-,A)$ exact
- $\mathrm{Ext}^i_R(B,A)=0$ for $i>0$ & any B.

— — Ext! — —

Ext also measures splitting of group extensions!

G is an ext. of B by A if

$$0 \to A \to G \to B \to 0$$

If it splits, $G \cong A \oplus B$.

$\mathrm{Ext}^1(A,B) = 0 \implies$ every ext. of B by A splits.