Homological Algebra Talk Prep

Commutative algebra reminders

→ **Lem. free $\implies$ projective $\implies$ flat $\implies$ torsion free**

Def. Flat R-module M

The functor $- \otimes_R M$ is exact; sends SES to SES.

Thm. 3.2.7 Every finitely presented flat R-module M is projective.

✓ **Ex.**

- $\mathbb{Z}/6\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ is projective but not free.

→ **Lem. nakayama's lemma**

Given a finitely generated A-module M, and $J(A)$ the Jacobson radical of A (intersection of all maximal ideals), with $I \subseteq J(A)$ some ideal, then if $IM = M$, we have $M = 0$.

Intuition: (M cannot be generated by such a small part of the whole ring, unless it is zero).

Q. How is the prop below a corollary of nakayama's lemma?

Prop. For a local ring A:

to verify that a function is non-zero everywhere on $Spec(A)$ it is enough to check that it is non-zero at a point, the unique closed point.

***Proof.* Let A be a local ring with unique maximal ideal m.**

Assume $f(m) \neq 0$ then $f \notin m \implies f \notin p \subseteq m$, then $f(p) \neq 0$ for all $p \in Spec(A)$.

Def. Direct sums and products (e.g. in Ab or R-Mod.)

Direct product $\prod_{\beta \in B} A_\beta := \{(a_\beta)_{\beta \in B}\}$
Direct sum $\bigoplus_{\beta \in B} A_\beta := \{(a_\beta)_{\beta \in B}$ | all but finitely many a_β are non-zero$\}$

Rem.

1. Direct sum and product coincide for $B \in$ Finset.
2. There is a formulation in terms of maps; map out of direct sums to map uniquely out of each A_β, map into direct products to map uniquely into each A_β.

Prop. For R-module M, where R is PID or local ring: M free $\iff$ M projective

Proof.

PID case:
Since for P projective we have some Q such that the universal map $P \oplus Q \to R$ is an isomorphism. Therefore, the generators of R are in bijection with the generators of $P \oplus Q$ and so the map $P \to R$ is an injective map of modules. All summands of a free module are submodules.

In a PID, we have no non-zero divisors, and all ideals are generated by one element. Let E be a submodule of R, then E is a principle ideal, generated by e. Then $h : R \to E; 1 \to e$ is a module homomorphism. Since R is an integral domain, h is injective, surjectivity comes from E being an ideal.
Therefore, $R \cong E$, and projective $\iff$ free.

Local ring case:
As above projective modules over a local ring R are submodules of R. If both P and Q are proper ideals then $P \oplus Q \subseteq m \subseteq R$ where m is the unique maximal ideal of R.

Therefore, at least one of P or Q contains an invertible element and so is all of R. The generators of $P \oplus Q$ and R are in bijection and so the other summand must be trivial, free of rank 0.

Def. perfect ring

R satisfies the descending chain condition on principle ideals

Def. If M is a left R-module R is a perfect ring then: M projective $\iff$ M flat

Def. Injective module

All maps $M \to I$ into injective module I factor through any injective morphism $M \hookrightarrow N$.

Def. Projective module

All maps $P \to M$ out of a projective module P factor through any surjective morphism $M \hookrightarrow N$.

Prop. Suppose R is a domain and M is an R-module. Then:

a) If M is injective, then M is divisible. In particular, Z is *not* an injective Z-module.
b) If M is divisible and torsionfree, then M is injective. In particular Q is an injective Z-module.
c) If R is a PID then every divisible module M is injective. More specifically if every element $a \in M$ can be 'divided' by any element of R, i.e. $\forall a \in M, r \in R$, $\exists b \in M\ a = br$.

Lem. (Injective Production Lemma)

If M is a flat R-module and N is an injective R-module, then $Hom_R(M, N)$ is injective.

Ex.

- $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.
- every injective abelian group is the direct product of some of $\mathbb{Q}, \mathbb{Z}[1/p]/Z$, e.g. $\mathbb{Q}/\mathbb{Z}$ (product of injectives is injective because Hom(A,-) commutes with products).
- The injective for A is the product of $Hom(A, Q/\mathbb{Z})$ copies of $\mathbb{Q}/\mathbb{Z}$.

Rem. $\mathbb{Q}/\mathbb{Z}$ is the direct limit of $\mathbb{Z}[1/p]$ $\forall p$

$\mathbb{Z}[1/1]/\mathbb{Z},\ \mathbb{Z}[1/2]/\mathbb{Z},\ \mathbb{Z}[1/3]/\mathbb{Z},\ \mathbb{Z}[1/5]/\mathbb{Z}, \ldots$
All include into $\mathbb{Q}/\mathbb{Z}$ in a unique way.

Prop. R-mod has enough injectives

Proof. **For R and integral domain,**

then mimic the $\mathbb{Q}/\mathbb{Z}$ construction with the fraction field of R.

See stacks project from Def. 15.55.5 onwards for the general proof.

AG ALERT

Prop. For ringed spaces (X, O_X) the category of sheaves of O_x-modules has enough injectives.

Proof.

For ring R, every R- module is iso to a submodule of an injective R-module.
Let F be a sheaf of O_x-modules.
Therefore, consider the sheaf J determined by $U \mapsto \prod_{x \in U} I_x$, for I_x the injective $O_{X,x}$-module of which F_x is a submodule.

Check that $Hom(-, J)$ is an exact functor, and that the sheaf morphism induced by $F_x \hookrightarrow I_x$ is also injective.

Ref. See Hartshorne AG chapter on sheaf cohomology.

Cor. The category of sheaves of abelian groups has enough injectives.

***Proof.* Let the ringed space (X, O_X) be the constant sheaf of rings $\mathbb{Z}$**

then the category of sheaves of $\mathbb{Z}$-modules is just the category of sheaves of abelian groups.

Category theory

Caution.

- Not all abelian categories have objects with elements:
- In this case it's necessary that our epimorphisms e are formulated in terms of maps ($f \circ e = g \circ e \implies f = g \, \forall f, g$),
- monomorphism m formulated in terms of maps ($m \circ f = m \circ g \implies f = g \, \forall f, g.$))
- Similarly for kernel, cokernel, and image, so $H_n := cok(ker(cok(d)) \xrightarrow{e} ker(d)$ (e is epimorphism so this can be thought of as a quotient, $im = ker \circ cok$).
- Diagram for kernel, and cokernel

Derived functors

Def. Left derived functor $L_i F$ of right exact functor F

The composite $H^\bullet \circ F \circ P$, where P is the pseudo-functor which takes you to an projective resolution of your object, then evaluate at F, then take homology.

Prop. This is well-defined

and (up to natural isomorphism) does not depend on the choice of projective resolution of A. p. 44 Wiebel

Proof. **Using the comparison Theorem**

Chain homotopic morphisms are equal under taking homology. (If $a \in ker(d_n) \implies d_n(a) = 0$ then $(f_n - g_n)(a) = (d_{n+1}s_n + s_{n-1}d_n)(a) = d_{n+1}s_n(a)$. So $d_{n+1}s_n(a) \in ker(d_n)$, but $d_{n+1}s_n(a) \in im(d_{n+1})$, so $H_n(f-g) = 0$. Hence, $H_n(f) = H_n(g)$.)

Assume P and Q are projective resolutions of A.
By the comparison theorem, lifts of id_A between projective resolutions are chain homotopic, functor F preserves chain homotopies, and so $H_n(F(f-g)) = H_n(F(f)) - H_n(F(g)) = 0$ and $L_n F := H_n(F(f))$ so the canonical chain map $f_* : (H_n(F(P)) \to H_n(F(Q)))_{n\in\mathbb{Z}}$.

Similarly, there is an induced canonical chain map $g_* : H_\bullet FQ \to H_\bullet FP$. Hence, by functoriality we have that $g_* f_* = (gf)_* = (id)_* = id_{H_\bullet F(P)}$ and by the same reasoning $f_* g_* = id_{H_\bullet F(Q)}$ so $H_\bullet(F(f))$ is a natural isomorphism.

Cor. If A is projective

Then $L_i(A) = 0$ for $i \neq 0$.

Proof.

$\ldots 0 \to 0 \to A \to 0$ is a projective resolution of A.

Thm. Each $L_i F$ is an additive functor.

Proof. **Exercise.**

Rem. Exact functors preserve derived functors

Because then almost by definition we get that the unique maps $F(ker(d)) \to ker(F(d))$ and $F(im(d)) \to im(F(d))$ give isomorphisms $F(H_n) \cong H_n(F)$.

Thm. derived functors $L_i F$ form homological δ-functors

Proof.

By horseshoe lemma for SES (A) and projective resolutions for A' and A'' (and by some definition of projective for P''), we get split exact sequences
$0 \to P_i' \to P_i \to P_i'' \to 0$
F is an additive functor and so preserves split exact sequences (we need them to be split in order to conclude that exactness is preserved).
So we get a short exact sequence of complexes which by the **snake lemma** induces a long exact sequence on homology.

See p.46 Wiebel to see that δ_i are natural transformations (uses horseshoe lemma and comparison lemma).

Showing derived functors are universal δ functors

Def. a functor T is effaceable if

for every object A in its domain we have a monomorphism $A \to D$ whose image under T. is the zero map.

Thm. For T a delta functor

if T^i is effaceable for all $i \geq 0$ then the functor is universal.

Proof.

($\implies$) Choose a delta functor for which all T^i are effaceable and a short exact sequence $0 \to I \to A \to B \to 0$ where $I \to A$ is the mono that is sent to zero under T_1.
So we have that $T^0(A) \to T^0(B) \to T^1(I) \to T^1(A) \to ...$ is exact and so $T^0(B) \to T^1(I)$ is surjective.

To determine f^1 we will use a general fact that for a map of exact sequences $[A \to B \to C \xrightarrow{\phi} 0] \mapsto [A' \to B', \to B' \to \dots]$ the maps $A \to A'$ and $B \to B'$ determine the map $(im(\phi) = B/A) \to (B'/A' \subseteq C')$.
Therefore, if ϕ is surjective then there is a unique $C \to C'$ such that the diagram commutes.

Then for any pair $(S, T^0 \xrightarrow{f^0} S^0)$ we have that the natural isomorphism maps $T^0 \to S^0$ completely determine $T^1 \xrightarrow{f^1} S^1$.

We proceed by induction to get a unique f^{i+1} from f^i.
Hence, T is universal.

Cor. Derived functors are universal δ-functors.

It follows then that under the assumption that A has enough injectives, that for a functor $A \xrightarrow{F} B$ of abelian categories $R^\bullet F$ is the universal δ functor where $T^0 \cong F$, $T^i := R^i F$.

Introducing Ext and Tor

Def. Adjunction

Functors $F : A \leftrightarrows B : G$ such that $Hom(F(a), b) \cong Hom(a, G(b))$

Ex. Tensor-Hom adjunction

$(- \otimes B) \dashv Hom(B, -)$

Try. Exercise for those not on the cat theory train

Prop. Category theory fact

Left adjoints preserve colimits (e.g. cokernel)
right adjoints preserve limits (e.g. kernel)

Rem. $Hom(-, B)$ is exact for B injective

***Proof.* By def of injective module, $Hom(-B)$ sends monos to monos.**

$Hom(-, B)$ sends epis to epis for any module B, but precomposing with a section of the surjection.

The dual statement for $Hom(P, -)$ is true for P projective.

Def. $Ext_\bullet$

For ring R and arbitrary R-module B, $Hom(B, -)$ is left exact and so has a **right derived functor** called $Ext_\bullet$ in R-mod.

Def. $Tor_\bullet$

For ring R and arbitrary R-module B, $(- \otimes B)$ is right exact and so has a **left derived functor** called $Tor_\bullet$ in R-mod.

Clarification.

- **Right** derived functors are computed using **projective** resolutions
- **Left** derived functors are computed using **injective** resolutions

Explanation

- **Both** $Hom(B, -) : C \to A$ **and** $Hom(-, B) : C^{op} \to A$ **are left exact** and so has a **right** derived functor Ext. However, projective resolutions are dual to injective resolutions, and so for the **contravariant Hom(-, B)** we use injective resolutions in the opposite category C^{op}, i.e. projective resolutions in C, whereas for covariant Hom(B, -) we use injective resolutions in C.
- $- \otimes B$ is left adjoint and so its left derived functor can be computed using projective resolutions.

Def. Sheaf cohomology

$\Gamma : Sh(V) \to R - mod$ is right adjoint to the constant sheaves functor, so Γ is left exact. **Sheaf cohomology is the right derived functor of** Γ.

Ex. For field k what is $R_\bullet Hom^{k[x]}(-, k)(k) = Ext_\bullet^{k[x]}(k, k)$

$k[x] \times k \to k; (f, a) \to f(0)a.$

1. Find projective resolution of k to substitute into the contravariant Hom (domain).
   Facts: $k[x]$ is PID so projective $\iff$ free.
   $0 \to k[x] \xrightarrow{x} k[x] \to k[x]/ < x > \to 0$
2. Then remove $k[x]/ < x >$ and take homology of
   $0 \to Hom_{k[x]}(k[x], k) \xrightarrow{\circ x = 0} Hom_{k[x]}(k[x], k) \to 0$
3. Work out $Hom_{k[x]}(k[x], k) = k$
4. $H_0 = k, H_1 = k \; H_i = 0$ for $i > 1$
5. $H_\bullet = k[x]/x^2$

✓ **Ex.**

$k[x]/x^2$ has the standard quotient module structure

1. proj resolution is $\ldots k[x] \xrightarrow{x^2} k[x] \to 0$
2. $\ldots Hom(k[x], k) \xrightarrow{-\circ x^2} Hom(k[x], k) \to 0$
3. $H_0 = k, H_1 =< x > / < x^2 >, H_i =< x^i > / < x^{i+1} >$
4. $H_\bullet = k[x]$

Prop. $k[x]$ **and** $k[x]/ < x^2 >$ **are koszul dual algebras.**

Useful properties in computing Ext

Thm. 2.7.6 For every pair of R-modules A and B, and all n,
$Ext^n_R(A, B) =^n Hom_R(A, -)(B) = R^n Hom_R(-, B)(A).$

Proof.

$Hom(A, I) \to Tot(Hom(P, I)) \leftarrow Hom(P, B)$ are quasi-isomorphisms.
So it doesn't matter if we compute using an injective resolution of A or a projective resolution of B, we get the same answer.
It is also possible to do both at the same time, and recover cohomology from a total

complex buy taking direct sums of the \ diagonals ($\oplus_{n=i+j} Hom(P_i, I^j)$).

$$
\begin{tikzcd}
\vdots & & & & \\
\mathrm{Hom}(A, I^2) \arrow[u] & \mathrm{Hom}(P_0, I^2) \arrow[u] \arrow[r] & \mathrm{Hom}(P_1, I^2) \arrow[u] \arrow[r] & \cdots & \\
\mathrm{Hom}(A, I^1) \arrow[u] & \mathrm{Hom}(P_0, I^1) \arrow[u] \arrow[r] & \mathrm{Hom}(P_1, I^1) \arrow[u] \arrow[r] & \mathrm{Hom}(P_2, I^1) \arrow[u] \arrow[r] & \cdots \\
\mathrm{Hom}(A, I^0) \arrow[u] & \mathrm{Hom}(P_0, I^0) \arrow[u] \arrow[r] & \mathrm{Hom}(P_1, I^0) \arrow[u] \arrow[r] & \mathrm{Hom}(P_2, I^0) \arrow[u] \arrow[r] & \cdots \\
 & \mathrm{Hom}(P_0, B) \arrow[r] & \mathrm{Hom}(P_1, B) \arrow[r] & \mathrm{Hom}(P_2, B) \arrow[r] & \cdots
\end{tikzcd}
$$

Prop. Ext and direct sums and products

$$\mathrm{Ext}^i_R\left(\bigoplus_\alpha M_\alpha, N\right) \cong \prod_\alpha \mathrm{Ext}^i_R(M_\alpha, N)$$

$$\mathrm{Ext}^i_R\left(M, \prod_\alpha N_\alpha\right) \cong \prod_\alpha \mathrm{Ext}^i_R(M, N_\alpha)$$

Prop. For all abelian groups A, B

- Tor_1(A,B) is a torsion abelian group
- Tor_n(A, B)=0 for $n \geq 2$

Tor Examples

Ex. $L^\bullet(\mathbb{Z}/5\mathbb{Z} \otimes_\mathbb{Z} B) =: Tor^\bullet_\mathbb{Z}(\mathbb{Z}/5\mathbb{Z}, B)$

1. projective resolution of $\mathbb{Z}/5\mathbb{Z}$:
   $0 \to \mathbb{Z} \xrightarrow{5} \mathbb{Z} \to \mathbb{Z}/5\mathbb{Z} \to 0$
2. Look at $0 \to \mathbb{Z} \otimes B \xrightarrow{5} \mathbb{Z} \otimes B \to \mathbb{Z}/5\mathbb{Z} \otimes B \to 0$
3. Find homology of $0 \to B \xrightarrow{5} B \to 0$
4. $H_0 := B/pB$, $H_1 := \{b \in B | 5b = 0\}$ gives us the 5-torsion elements of B.

Prop. B a flat R-module implies $Tor^R_n(A, B) = 0 \; \forall n \neq 0$

Proof.

If we take a projective resolution $P_\bullet$ of A, then $P_\bullet$ is exact in non-zero degrees by construction and since B is flat $P_\bullet \otimes B$ is exact in degrees $n \geq 1$.

Lem.

$R = Z, B = Q$ shows that flat modules need not be projective.

Prop. If R is principal ideal domain then an R-module B is flat $\iff$ B is torsion-free.

Ex. So in our example, if B is torsion free, then higher tor groups are trivial.

Ex. $Tor_\bullet^{k[x,y]}(k,k)$ probe the failure of k to be flat as a $k[x,y]$-module

1. projective resolution $0 \to k[x,y] \xrightarrow{[-y,x]} k[x,y]^2 \xrightarrow{[x,y]} k[x,y] \to k \to 0$

   Try. tensor with k then find cohomology.

Caution.

- Not all abelian categories have enough injectives or projectives but we can still define left and right derived functors due to **Yoneda Lemma** and **Baer's sum**.

Balancing

Rem. Left derived functors are right derived functors on the opposite categories

and so all results apply that apply to left, also apply to right.

Thm. Yoneda

Intuition: an object in completely determined by all maps in/out of it; $(Hom(A,X))_{A\in C}$ determines X.
Statement: $Hom_{Psh(C)}(H_A, Y) \cong Y(A)$
$Hom_{Psh}(H_1, c_{R(B)}) \cong R(B)$
$H_1(A) \cong A$, and $c_{R(B)}(A) = R(B)$, so we get all maps $(A \to R(B))$.

↳ Cor. Adjoints

Left adjoints are right exact and therefore when the codomain has enough projectives they have left derived functors, right adjoints are left exact and therefore when their codomains have enough injectives they have right derived functors.

Alt. *Proof.*

Apply adjunction natural iso to short exact sequence, then apply **Yoneda** to $0 \to Hom(A, R(B')) \to Hom(A, R(B)) \to Hom(A, R(B'')) \to 0$ to get exactness of $R(B_\bullet)$.

✓ Ex. So in some cases we can get Tor from Ext.

...

Filtered colimits and direct limits

→ Lem. 2.6.14 Let Z be a filtered category and $A : I \to mod - R$ a functor.

Then

1. Every element $a \in coim(A_i)$ is the image of some element $ai \in A_i$ (for some $i \in Z$) under the canonical map $A_i \to colim(A_i)$.
2. For every i, the kernel of the canonical map $A_i \to colim(A_i)$ is the union of the kernels of the maps $\phi : A_i \to A_j$ (where $\phi : i \to j$ is a map in Z).

Thm. 2.6.15 Filtered colimits (and direct limits) of R-modules are exact, considered as functors from (mod-R) to mod-R.

A short exact sequence of filtered colimit diagrams induces a short exact sequence of colimits.

Ex.