# Mapping Cones

Let $f : B_\bullet \to C_\bullet$ be a map of chain complexes

**Def$^n$**: The mapping cone of $f$ is the chain complex

deg $n$ part is $B_{n-1} \oplus C_n$

$$\mathrm{cone}(f) := \quad \cdots \to \begin{matrix} B_n \\ \oplus \\ C_{n+1} \end{matrix} \to \begin{matrix} B_{n-1} \\ \oplus \\ C_n \end{matrix} \to \begin{matrix} B_{n-2} \\ \oplus \\ C_{n-1} \end{matrix} \to \cdots$$

with the differential

$$d_n : \begin{matrix} B_{n-1} \\ \oplus \\ C_n \end{matrix} \longrightarrow \begin{matrix} B_{n-2} \\ \oplus \\ C_{n-1} \end{matrix}$$

$$\begin{pmatrix} b \\ c \end{pmatrix} \longmapsto \begin{pmatrix} -d_B(b) \\ d_C(c) - f(b) \end{pmatrix}$$

given by the matrix

$$\begin{pmatrix} -d_B & 0 \\ -f & d_C \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix} = \begin{pmatrix} -d_B(b) \\ d_C(c) - f(b) \end{pmatrix}$$

The dual notion applies to a map $f : B^\bullet \to C^\bullet$ of cochain complexes, with cone

$$\cdots \to \begin{matrix} B^{n+2} \\ \oplus \\ C^{n+1} \end{matrix} \to \begin{matrix} B^{n+1} \\ \oplus \\ C^n \end{matrix} \to \begin{matrix} B^n \\ \oplus \\ C^{n-1} \end{matrix} \to \cdots$$

and the same differential.

Last week:

**Thm:** For S.E.S. $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$, there are

chain complexes

connecting homomorphisms $\partial : H_n(C) \to H_{n-1}(A)$ s.t.

$$\cdots \xrightarrow{g} H_{n+1}(C) \xrightarrow{\delta} H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{f} \cdots$$

is an exact sequence.

We can fit $f_*$ : $H_*(B) \to H_*(C)$ into a L.E.S. using the S.E.S. of chain complexes

the push forward

$$c \longmapsto \begin{pmatrix} 0 \\ c \end{pmatrix}$$

$$0 \to C_\bullet \to \operatorname{cone}(f) \xrightarrow{\delta} B_\bullet[-1] \to 0.$$

$$\begin{pmatrix} b \\ c \end{pmatrix} \longrightarrow \begin{pmatrix} -b \\ 0 \end{pmatrix}$$

Then since $H_{n+1}(B[-1]) \cong H_n(B)$, (1.2.8)

we have homology L.E.S. given by

$$\cdots \longrightarrow H_{n+1}(\operatorname{cone}(f)) \xrightarrow{\delta} H_n(B) \xrightarrow{\partial} H_n(C) \longrightarrow H_n(\operatorname{cone}(f)) \xrightarrow{\delta} H_{n-1}(B) \xrightarrow{\partial} H_{n-1}(C) \longrightarrow \cdots$$

We can see that $f_*$ is the map $\partial$ here, since for $b \in B_n$, a cycle (in $\ker d$)

$$b \xrightarrow{\delta} \begin{pmatrix} -b \\ 0 \end{pmatrix} \xrightarrow{\begin{pmatrix} -d_B & 0 \\ -f & d_C \end{pmatrix}} \begin{pmatrix} db \\ fb \end{pmatrix}$$

$db = 0$, since $b$ is a cycle and therefore in the kernel of $d$.

$\partial[b] = [fb] = f_*[b]$

Motivation: Can think of mapping cone as analyzing $f$ by collapsing $B$ into a simpler structure glued to $Y$.

Cor$^y$: $f : B \to C$ is a quasi-isomorphism

$$\Updownarrow$$

$\operatorname{cone}(f)$ is exact.

"Mapping cone" is closely related to "mapping cylinder"

Def$^n$: The mapping cylinder of $f$ is the chain complex

$$\operatorname{cyl}(f) := \quad \cdots \longrightarrow \begin{matrix} B_{n+1} \\ \oplus \\ B_n \\ \oplus \\ C_{n+1} \end{matrix} \longrightarrow \begin{matrix} B_n \\ \oplus \\ B_{n-1} \\ \oplus \\ C_n \end{matrix} \longrightarrow \begin{matrix} B_{n-1} \\ \oplus \\ B_{n-2} \\ \oplus \\ C_{n-1} \end{matrix} \longrightarrow \cdots$$

with differential

$$d : \begin{matrix} B_n \\ \oplus \\ B_{n-1} \\ \oplus \\ C_n \end{matrix} \longrightarrow \begin{matrix} B_{n-1} \\ \oplus \\ B_{n-2} \\ \oplus \\ C_{n-1} \end{matrix}$$

$$\begin{pmatrix} b \\ b' \\ c \end{pmatrix} \longmapsto \begin{pmatrix} d_B & id_B & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{pmatrix} \begin{pmatrix} b \\ b' \\ c \end{pmatrix}$$

Here, we have a chain complex since $d^2 = 0$
(square the matrix to check)

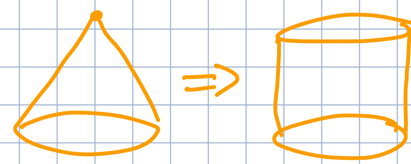
By a similar method as before, we can use the S.E.S.

$$B \to \mathrm{cyl}(f) \xrightarrow{\beta} C$$

to induce (via steps detailed in Weibel) the L.E.S.

$$\begin{tikzcd}
\cdots \arrow[r, "-\partial"] & H_n(B) \arrow[r] \arrow[d, equal, "\cong"] \arrow[dr, "f"] & H_n(\mathrm{cyl}(f)) \arrow[r] \arrow[d, equal, "\cong"] & H_n(\mathrm{cone}(f)) \arrow[r, "-\partial"] \arrow[d, equal] & H_{n-1}(B) \arrow[r] \arrow[d, equal, "\cong"] \arrow[dr, "f"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(B[-1]) \arrow[r, "\partial"'] & H_n(C) \arrow[r] & H_n(\mathrm{cone}(f)) \arrow[r, "\delta"'] & H_n(B[-1]) \arrow[r, "\partial"'] & \cdots
\end{tikzcd}$$

the L.E.S. we obtained last time



**Lemma:** This diagram commutes and has exact rows.

$\forall$ chain map $f: B \to C$, the cone and cylinder constructions provide a natural way to fit the homology of $f$ into some exact sequence as shown.
(To see that this L.E.S. is always well defined, see W. p23)

# Abelian Categories

- based on 2024 SMSTC Homological Algebra lecture notes

## Some definitions:

For $\mathcal{C}$ a category, $f : A \to B \in \mathrm{Mor}(\mathcal{C})$ is

(i) a monomorphism if for all constructions

$$\begin{tikzcd} C \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

we have

$$f \circ g = f \circ h \implies g = h$$

(ii) an epimorphism if $\forall$ constructions

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

we have

$$g \circ f = h \circ f \implies g = h$$

For $\mathcal{C}$ a category,

(i) an initial object is $I \in \mathrm{Ob}(\mathcal{C})$ s.t. $\forall X \in \mathrm{Ob}(\mathcal{C})$ $\exists!$ $I \to X \in \mathrm{Mor}(\mathcal{C})$

$$\begin{tikzcd} & X \\ I \arrow[ur, "\exists!"] \arrow[r, "\exists!"] \arrow[dr, "\exists!"] & Y \\ & Z \end{tikzcd}$$

(ii) a terminal object is $T \in \mathrm{Ob}(\mathcal{C})$ s.t. $\forall X \in \mathrm{Ob}(\mathcal{C})$ $\exists!$ $X \to I \in \mathrm{Mor}(\mathcal{C})$

$$\begin{tikzcd} X \arrow[dr, "\exists!"] & \\ Y \arrow[r, "\exists!"] & T \\ Z \arrow[ur, "\exists!"] & \end{tikzcd}$$

(iii) A **zero object** is both initial and terminal.

Ab - category in Weibel

**Def$^n$:** $\mathcal{C}$ is **preadditive** if all hom-sets are abelian groups and we have bilinear composition of morphisms.

$$(g+h)\circ f = g\circ f + h\circ f$$
$$f\circ(g+h) = f\circ g + f\circ h$$

Last week: $\ker(f) = \{a \in A \mid f(a) = 0\}$

$$\operatorname{coker}(f) = B/\operatorname{im}(f)$$

**Product** $A\times B$ is s.t. $\forall$ constructions

$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, dashed, "\exists!u"] & \\
A & A\times B \arrow[l, "pr_1"] \arrow[r, "pr_2"'] & B
\end{tikzcd}$$

$A\times B$ is the object with universal property shown in blue.

**Coproduct** $A\oplus B$ is s.t. $\forall$ constructions

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[dr, "s"'] & A\oplus B \arrow[d, dashed, "\exists!v"] & B \arrow[l, hook'] \arrow[dl, "t"] \\
 & C &
\end{tikzcd}$$

$A\oplus B$ is the object with universal property shown in blue.

**Def$^n$:** A preadditive category is <u>additive</u> if it admits finite products and has a zero object.

functor

**Def$^n$:** A functor $F : \mathcal{A} \to \mathcal{B}$ on preadditive $\mathcal{A}$ and $\mathcal{B}$ with finite products is <u>additive</u> if

1) $\forall X, Y \in Ob(\mathcal{A})$, $F : Hom_{\mathcal{A}}(X,Y) \longrightarrow Hom_{\mathcal{B}}(F(X), F(Y))$ is a group homomorphism, and

2) $\forall X, Y \in Ob(\mathcal{A})$, $F(X \oplus Y) \cong F(X) \oplus F(Y)$

Special type of additive category:

**Def$^n$:** A category is <u>abelian</u> if

- it is preadditive
- it has zero object
- it has all binary products and coproducts
- it has all kernels and cokernels
- every monomorphism is the kernel of its own cokernel
- every epimorphism is the cokernel of its kernel

# Derived functors

Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories.

**Def$^n$:** A homological $\delta$-functor between $\mathcal{A}$ and $\mathcal{B}$ is a collection of additive functors

$$(T_n : \mathcal{A} \to \mathcal{B})_{n \geq 0}$$

with "shift morphisms" defined on a S.E.S. $0 \to A \to B \to C \to 0 \in \mathcal{A}$:

$$\delta_n : T_n(C) \to T_{n-1}(A)$$

s.t.

1) $\exists$ L.E.S.

$$\cdots T_{n+1}(C) \xrightarrow{\delta} T_n(A) \to T_n(B) \to T_n(C) \xrightarrow{\delta} T_{n-1}(A) \to \cdots$$

2) $\forall$ morphism

$$\begin{array}{c} 0 \to A' \to B' \to C' \to 0 \\ \downarrow f \\ 0 \to A \to B \to C \to 0 \end{array}$$

the diagram

$$\begin{tikzcd} T_n(C') \arrow[r, "\delta"] \arrow[d, "f"'] & T_{n-1}(A') \arrow[d, "f"] \\ T_n(C) \arrow[r, "\delta"] & T_n(A) \end{tikzcd}$$

commutes.

Cohomology: flip arrows in the exact sequences, and use superscript

**Example:** Let $p \in \mathbb{Z}$

$$T_0(A) = A/pA \qquad \text{(cokernel)}$$

$$T_1(A) = \{a \in A : pa = 0\} =: {}_pA \qquad \text{(kernel)}$$

$\exists$ a $\delta$-functor from $Ab$ to $Ab$:

Apply Snake lemma to

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "p"] & B \arrow[r] \arrow[d, "p"] & C \arrow[r] \arrow[d, "p"] & 0 & \in Ab \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 & \in Ab
\end{tikzcd}$$

to get exact sequence

$$0 \to {}_pA \to {}_pB \to {}_pC \xrightarrow{\delta} A/pA \to B/pB \to C/pC \to 0$$

**Def$^n$:** A <u>morphism of $\delta$-functors</u> $S \to T$ is a system of natural transformations

$$(S_n \to T_n)_{n \in \mathbb{N}}$$

that commute with $\delta$.

I.e. $\forall$ S.E.S. in $\mathcal{A}$, the associated L.E.S. in $S$ and L.E.S. in $T$ are connected by a commutative ladder diagram

$$0 \to A \to B \to C \to 0$$

$S$ $\qquad$ $T$

$$\begin{tikzcd}
\cdots & S_{n+1}(C) \arrow[r, "\delta"] \arrow[d] & S_n(A) \arrow[r] \arrow[d] & S_n(B) \arrow[r] \arrow[d] & S_n(C) \arrow[r, "\delta"] \arrow[d] & S_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
\cdots & T_{n+1}(C) \arrow[r, "\delta"] & T_n(A) \arrow[r] & T_n(B) \arrow[r] & T_n(C) \arrow[r, "\delta"] & T_{n-1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

**Def$^n$:** A $\delta$-functor $T$ is <u>universal</u> if, given any other $\delta$-functor $S$ and $0^{th}$ natural transformation $f_0 : S_0 \to T_0$, $\exists$ unique morphism

$$\{f_n : S_n \to T_n\}$$

of $\delta$-functors extending $f_0$.

# Projective resolutions

Let $\mathcal{A}$ be an abelian category.

**Def$^n$**: $P$ is *projective* if:

$$\forall \begin{cases} \text{surjections} & g : B \twoheadrightarrow C \\ \text{maps} & \gamma : P \longrightarrow C \end{cases}$$

$$\exists \; \beta : P \to B \quad \text{s.t.} \quad \gamma = g \circ \beta$$

$$\begin{tikzcd} & P \arrow[d, "\gamma"] \arrow[dl, "\exists \beta"'] & \\ B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

a given ring

Let $\mathcal{A}$ be the category mod-$R$.

**Prop$^n$**: An $R$-module is projective iff it is a direct summand of a free $R$-module.

**Def$^n$**: $\mathcal{A}$ has *enough projectives* if $\forall A \in Ob(\mathcal{A})$, there is a *surjective projective*

$$P \twoheadrightarrow A.$$

A chain complex of projectives is $P := \cdots \to P_{n+1} \to P_n \to P_{n-1} \to \cdots$ where each $P_n$ is projective. $P$ needn't be a projective object in $Ch$.

If $\mathcal{A}$ has enough projectives, $Ch(\mathcal{A})$ does too.

category of chain complexes over $\mathcal{A}$.

**Def$^n$**: Let $M \in Ob(\mathcal{A})$.

A *left resolution of $M$* is a complex $P_\bullet$ with a map $\varepsilon : P_0 \to M$ so that

$$\underbrace{\cdots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\varepsilon} M \to 0}_{\text{augmented complex}}$$

is exact.

$P_\bullet \to M \to 0$ in short.

It is a *projective resolution* if the $P_i$ are all projective.

**Lemma**: If an abelian category $\mathcal{A}$ has e.p., then every object $M \in Ob(\mathcal{A})$ has a projective resolution.

$$\begin{tikzcd}
0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 \\
& M_3 \arrow[ur] \arrow[dr] & & & & M_1 \arrow[ur] \arrow[dr] \\
\cdots \arrow[ur] \arrow[rr] & & P_3 \arrow[rr, "d"] \arrow[dr] & & P_2 \arrow[rr, "d"] \arrow[ur] & & P_1 \arrow[rr, "d"] \arrow[dr] & & P_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0 \\
& & & M_2 \arrow[ur] \arrow[dr] & & & & M_0 \arrow[ur] \arrow[dr] \\
& & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0
\end{tikzcd}$$

- $(P_0, \varepsilon)$ exists since e.p.
- Define $M_i$ and $P_i$ inductively to make it exact.

**Comparison Theorem**:

Let $P_\bullet \xrightarrow{\varepsilon} M$ be a projective resolution of $M$.
Let $f' : M \to N$ a map in $\mathcal{A}$.
Then $\forall$ resolution $Q_\bullet \xrightarrow{\eta} N$ of $N$, $\exists!$ (up to chain homotopy equivalence) a chain map $f : P_\bullet \to Q_\bullet$

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "\exists f_2"'] & P_1 \arrow[r] \arrow[d, "\exists f_1"'] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "\exists f_0"'] & M \arrow[r] \arrow[d, "f'"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0
\end{tikzcd}$$

s.t. $\eta \circ f_0 = f' \circ \varepsilon$.

In general we only need $P_\bullet \to M$ to be a chain complex with $P_i$ projective (not exact), but we will use this stronger version in order to construct the external product for Tor.

## Horseshoe Lemma:

For a commutative diagram

exact

proj. res.

proj. res.

$$\begin{tikzcd}
 & & & & 0 \arrow[d] \\
\cdots \arrow[r] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r] & A' \arrow[d, "i_A"] \\
 & & & & A \arrow[d, "\pi_A"] \\
\cdots \arrow[r] & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r] & A'' \arrow[d] \\
 & & & & 0
\end{tikzcd}$$

then the collection $\{P_n := P_n' \oplus P_n''\}_{n\in\mathbb{N}}$ form a proj. res. $P_\bullet$ of $A$, and the columns are exact sequences

$$0 \to P' \xrightarrow{i} P \xrightarrow{\pi} P'' \to 0$$

(We can fill in the Horseshoe in the way we'd like)

# Injective Resolutions

Let $\mathcal{A}$ be an abelian category.

**Def$^n$**: $I$ is *injective* if:

$$\forall \begin{cases} \text{injections} & f : A \hookrightarrow B \\ \text{maps} & \alpha : A \to I \end{cases}$$

$$\exists \ \beta : B \to I \quad \text{s.t.} \quad \alpha = \beta \circ f$$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[dl, dashed, "\exists \beta"] \\ & I & \end{tikzcd}$$

R. Baer

inj modules were invented in 1940
long before proj modules

An abelian category $\mathcal{A}$ has *enough injectives* if $\forall A \in Ob(\mathcal{A})$ there is an
injective injective

$$A \hookrightarrow I$$

**Baer's Criterion**: A right $R$-module $E$ is injective

$$\Updownarrow$$

$$\forall \begin{cases} \text{right ideal } J \triangleleft R, \\ \text{map } j : J \to E \end{cases}$$

$$\exists \text{ an extension } j' : R \to E$$

$$\begin{tikzcd} & & E \\ R \arrow[urr, dashed, "\exists"] \arrow[r, phantom, "\triangleright"] & & J \arrow[u] \end{tikzcd}$$

**Lemma**: Let $\mathcal{A}$ be an abelian category,
Let $I \in Ob(\mathcal{A})$.
TFAE:

- $I$ is injective in $\mathcal{A}$
- $I$ is projective in $\mathcal{A}^{op}$
- $Hom_{\mathcal{A}}(-, I)$ is exact

**Def$^n$:** Let $M \in Ob(\mathcal{A})$.

A right resolution of $M$ is a complex $P_\bullet$ with a map $\varepsilon : M \to I^0$ so that

$$\underbrace{0 \longrightarrow M \xrightarrow{\varepsilon} I^0}\ \underbrace{\xrightarrow{d} I^1 \xrightarrow{d} I^2 \xrightarrow{d} \cdots}$$

augmented complex

is exact.

It is an injective resolution if the $I^i$ are all projective.

**Lemma:** Abelian cat $\mathcal{A}$ has enough injectives

$\Rightarrow$ every object in $\mathcal{A}$ has an inj. res.

**Comparison Theorem:**

Same for injectives as for injectives, except (the arrows flipped, the "co" added, the indexes superscripted) the inj. resolution we start with is the bottom row, with all other resolutions mapping "to it" not from it.

**Lemma:** $\forall$ right $R$-module $M$, the natural map

$$\boxed{\tau : Hom_{Ab}(M, A) \longrightarrow Hom_{R\text{-mod}}(M, Hom_{Ab}(R, A))}$$

$$\begin{tikzcd} & R \arrow[d] \\ M \arrow[ur, bend left, "\text{morphism of } R\text{-modules}"] \arrow[r, "\text{morphism of } Ab"'] & A \end{tikzcd}$$

is an isom, where $f : M \to A$ has

$$\tau f : \underset{\in M}{m} \longmapsto \left( \begin{array}{c} r \in R \\ \downarrow \\ f(mr) \in A \end{array} \right)$$

**Lemma:** If $I$ is an injective abelian group then $\mathrm{Hom}_{Ab}(R, I)$ is an injective $R$-module

**Example:** $Sh(X)$ (category of abelian group sheaves on top space $X$) has e.i.

The skyscraper sheaf

$$(x_* A)(\mathcal{U}) = \begin{cases} A & \text{if } x \in \mathcal{U} \\ 0 & \text{otherwise,} \end{cases}$$

and the stalk

$$\mathcal{F}_x = \lim \{ \mathcal{F}(\mathcal{U}) : x \in \mathcal{U} \}$$

give $\quad \mathrm{Hom}_{Ab}(\mathcal{F}_x, A) \cong \mathrm{Hom}_{Sh(X)}(\mathcal{F}, x_* A).$

Then $\forall$ sheaf $\mathcal{F}$, we have injective injective (in Ab)

$$\mathcal{F}_x \longrightarrow I_x$$

$\forall x \in X$ which induces an injection

$$\mathcal{F} \hookrightarrow I = \prod_{x \in X} x_*(I_x).$$

Then $Sh(X)$ has e.i.