Reading Seminar on *Homological Algebra*

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1 Sequences

Ler R will be a ring, and we will work in the R-modules category.

Definition 1.1. Sequence
A sequence of R-module is a collection $(A_n, f_n)_{n\in\mathbb{Z}}$ such that $f_n : A_n \to A_{n-1}$.

The dual construction is called the **co-sequence** and it is defined as follows:

Definition 1.2. Co-sequence A co-sequence of R-modules is a collection of $(A_n, f_n)_{n\in\mathbb{Z}}$ such that $f_n : A_n \to A_{n+1}$.

We can apply these functions to a chain of R-modules and we get the concept of an exact sequence:

Definition 1.3. Exact sequence
$... \to A_n \to A_{n-1} \to A_{n-2}...$ is exact at A_n if $\Im f_{n+1} = \ker f_n$. If it is exact at every term it is called an **exact sequence**.

Let us take an example: The sequence $0 \to A \to B \to 0$ is

1. exact at $A \iff f$ is injective;
2. exact at $A \iff f$ is surjective.

Therefore, $0 \to A \to B \to 0$ is exact if and only if f is an isomorphism.
We can see that there are nicer sequences. This brings us to the concept of a **short exact sequence**.

Definition 1.4. Short exact sequence A short exact sequence is an exact sequence of the form $0 \to A \to B \to C \to 0$, where $B \cong B/\Im f = B/\ker g$.

Remark 1.5. A short exact sequence is called split if $B = A \oplus B$.

Definition 1.6. Long exact sequence A long exact sequence

$$A_0 \to A_1 \to A_2... \to A_n$$

is a family of SES for $K_i = \Im f_i$,

$$\begin{gathered} 0 \to K_1 \to A_1 \to K_2 \to 0 \\ 0 \to K_2 \to A_2 \to K_3 \to 0 \\ 0 \to K_3 \to A_3 \to K_4 \to 0 \\ ...\ 0 \to K_{n-1} \to A_{n-1} \to K_n \to 0 \end{gathered}$$

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Definition 1.7. Morphism of sequences Let $\{A_n\}, \{B_n\}$ be sequences of R-modules. A morphism from these sequences is a collection of maps $A_n \to B_n$ such that the following diagram commutes:

$$... \to A_{n+1} \to A_n \to A_{n-1} \to ...$$

$$... \to B_{n+1} \to B_n \to B_{n-1} \to ...$$

that is $du = ud$.

Remark 1.8. Sequences form a category with these morphisms where the identity morphism is given by the identity map on $A_n \to A_n$.

2 Chain complexes

We have seen what sequences are, now let us take an example from physics and differential calculus:

- smooth functions $:= C^{\infty}(\mathbb{R}^3)$;
- vector fields $:= F(\mathbb{R}^3)$.

Let us take the following sequences

$$C^{\infty}(\mathbb{R}^3) \to F(\mathbb{R}^3) \to F(\mathbb{R}^3) \to C^{\infty}(\mathbb{R}^3),$$

where **grad** takes a scalar field into a vector, **curl** takes a vector into a vector, and **div** takes a vector into a scalar. For a function f and a vector field X we have

1. curl (grad f) $= 0$ (steepest ascent doesn't spin around);
2. div (curl$X$) $= 0$.

Both of these results are true because partial derivatives commute $\to$ much of differential calculus is about some kind of vector spaces "chained" together by linear maps whose successive compositions are 0.

Definition 2.1. Chain complex A chain complex $C.$ of R-modules is a sequence $\{C_n\}_{n\in\mathbb{Z}}$ such that the maps $d = d_n : C_n \to C_{n-1}$ are called **differentials** and $d^2 : C_n \to C_{n-2} = 0$.

On the other hand, we can define its dual structure, the cochain complex.

Definition 2.2. A cochain complex $C^{\cdot}$ of R-modules is a sequence $\{C_n\}_{n\in\mathbb{Z}}$ such that the maps $d = d_n : C_n \to C_{n+1}$ are called **differentials** and $d^2 : C_n \to C_{n+2} = 0$.

Since we defined morphisms on sequences, morphisms of chain complexes are called **chain maps**.

Remark 2.3. Operation on chain complexes
Translation: $C[p]_n = C_{n+p}$ for $p \in \mathbb{Z}$;

(CO)-HOMOLOGY

$$... \to A_{n+1} \to A_n \to A_{n-1} \to ...$$

Since the double differential is 0, we can conclude that $\Im d_{n+1} \subseteq \ker d_n, \forall n,$, therefore, we have a well-defined quotient module $\ker d_n/\Im d_{n+1} = H_n(C.)$ that we call the **n th Homology of the chain complex C.** .

To get more formal with the definition of this structure, we define the cycle and the boundary.

Definition 2.4. n-cycle for $c \in C_n, d_n c = 0$. The kernel of d_n is the modules of n-cycles of $C.$ that are noted as $Z_n(C.)$.

Definition 2.5. n-boundary for $c \in C_n, c = d_{n+1}c'$ for $c' \in C_{n+1}$. The image of d_{n+1} is the module of n-boundaries noted as $B_n(C.)$.

Therefore we can write the homology as $Z_n(C.)/B_n(C.)$.

Similar definitions for cocycles and coboundaries, therefore we can define the **cohomology** $= H^n(C^\cdot) = Z^n(C^\cdot)/B^n(C^\cdot)$.

Remark 2.6. Given a morphism of maps, there is an induced homomorphism from their respective homologies called the **induced homomorphism**.

Definition 2.7. Quasi-isomoprhism A chain map $C. \to D.$ is called a quasi-isomorphism if all the induced homomorphisms are isomorphisms.

Remark 2.8. The relation 'There exists quasi-isomorphism from $C.$ to $D.$' is not an equivalence relation since it s not transitive!
Its transitive closure allows us to define the **derived category of chain complexes**.

3 Snake lemmas

Lemmas defined over an Abelian Category that is a category where every hom-set in this category is given the structure of an Abelian group such that the composition distributes over addition.

Example 3.1. The category of R-modules is an Abelian category.

Theorem 3.2. *1st Snake Lemma Let A be an abelian category and suppose that we have a diagram of the form:*

$$A \to B \to C \to 0$$

$$0 \to A' \to B' \to C',$$

where the rows are exact. Then, there is a morphism $\delta : \ker a \to cokerc$, $cokera = A'/\Im a$, such that the following sequence is exact:

$$\ker a \to \ker b \to \ker c \to cokera \to cokerb \to cokerc.$$

Proof. We will discuss the case for A being the cat of R-modules. The case for abelian categories can be done using the Embedding Theorem that allows us to apply the theorems developed in the case of Mod-categories in the case of Abelian categories.

The proof of this lemma is done via 'diagram chasing'.
Let us take an element $z \in C$ such that $c(z) = 0$. This means that $z \in \ker c$. Since g is onto, $\exists y \in B$ so that $z = g(y)$, therefore we have by applying this process to both parts that $g'b(y) = cg(y) = c(z) = 0$, therefore $b(y) \in \ker g' = \Im f'$, hence exists $x' \in A'$ such that $f'(x') = b(y)$.
Put $\delta(x) = x' + \Im a \in \text{coker}a$. We prove that this map is

1. well-defined;
   Pick $y_1 \in B$ such that $g(y_1) = z$, then $y - y_1 \in \Im f = f(x)$ for some x. Let $x_1^{'}$ such that $f'(x_1^{'}) = g(y_1)$, then $f'(x' - x_1^{'}) = b(y - y_1) = bf(x) = f'(a(x))$. Therefore, $a(x) = x' - x_1^{'}$ since f' is injective and $x' + \Im a = x_1^{'} + m\Im a$ as elements in the cokernel of a.

2. homomorphism.
   Let $z_1, z_2 \in \ker c$, and for each i, let $(x_, y_i)$ be the pair in the def. of δx_i, then $g(y_1 - y_2) = z_1 - z_2$ and $f'(x_1 - x_2) = b(y_1 - y_2)$ so $\delta(z_1 - z_2) = x_1 - x_2$. In the same manner we do for $\delta(rx_1) = r\delta(x_1)$.

□

Theorem 3.3. *2nd Snake Lemma Let $0 \to A. \to B. \to C. \to 0$ be a short sequence of chain complexes, then there exists natural maps $\delta : H_n(C) \to H_{n-1}(A)$ called* ***connecting homomorphism*** *such that:*

$$\ldots \to H_{n+1}(C) \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to \ldots$$

is an exact sequence.

Proof. ☐

1. Form the following sequence as in the book.

2. Take each sequence and apply Snake Lemma.

3. Figure out the kernel and cokernel.

4 Chain homotopy

We will compare this concept with the topological homotopy which is an equivalence relation of maps. We define this new concept using chain complexes.

Definition 4.1. Chain homotopy Let $f, g : C. \to D.$ be chain maps. A chain homotopy form f to gn is a collection of maps $s_n : C_n \to D_{n+1}$ such that for each n we have:

$$f_n - g_n = s_{n-1}d_n - d_{n+1}s_n.$$

If there is a chain homotopy from f to g we write $f \cong g$.

Here is the diagram:

We can observe that as in the topological case, this relation $\cong$ is an equivalence relation on chain maps.

Lemma 4.2. *The relation $\cong$ is an equivalence relation on chain maps.*

Proof. We check the three properties:

1. reflexivity: take $s_n = 0$;
2. symmetry: if s is a chain homotopy from f to g, then $-s$ is a chain homotopy from g to f;
3. transitivity: if h is a chain homotopy from f to g, h' is a chain homotopy from g to k, then $h + h'$ is a chain homotopy from f to k.

□

Definition 4.3. Chain homotopy equivalence We say that $f : C. \to D.$ is a chain homotopy equivalence if there is a mag $g : D. \to C.$ such that gf and fg are chain homotopic to their respective identity maps on $C.$ and $D.$.

Proposition 4.4. *Let $C.$ and $D.$ be two chain complexes and $f, g : C. \to D.$ chain homotopic. Then, the induced maps on the homology are equivalents:*