Leray

DEF: a double complex $A^{\bullet,\bullet}$ is a system of objects $A^{i,j}$ together with maps:

$d_h^{i,j} : A^{i,j} \longrightarrow A^{i+1,j}$ horizontal diff.

$d_v^{i,j} : A^{i,j} \longrightarrow A^{i,j+1}$ vertical diff.

s.t.

$$d_h^{i+1,j} \circ d_h^{i,j} = 0$$

$$d_v^{i,j+1} \circ d_v^{i,j} = 0$$

$$d_v^{i+1,j} \circ d_h^{i,j} + d_h^{i,j+1} \circ d_v^{i,j} = 0$$

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & A^{i,j+2} \arrow[u] \arrow[r] & A^{i+1,j+2} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & A^{i,j+1} \arrow[u] \arrow[r] & A^{i+1,j+1} \arrow[u] & \cdots \\
\cdots \arrow[r] & A^{i,j} \arrow[u] \arrow[r] & A^{i+1,j} \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots &
\end{tikzcd}$$

$m \in \mathbb{Z}$

$$A_m = \bigoplus_{i+j=m} A^{i,j}$$

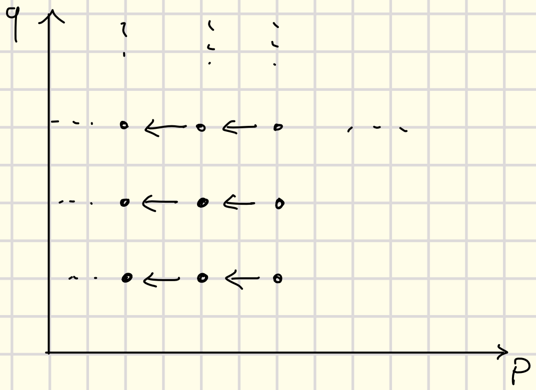
DEF: a HOMOLOGY SPECTRAL SEQUENCE (HSS) in an abelian category is given by the following data:

- $\{E^r_{p,q}\}$ objects with $p,q \in \mathbb{Z}$ and $r \geq a$
- maps $d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$ s.t. $d \circ d = 0$
- isomorphism between $E^{r+1}_{p,q} \cong \operatorname{Ker}(d^r_{p,q}) / \operatorname{Im}(d^r_{p+r,q-r+1})$

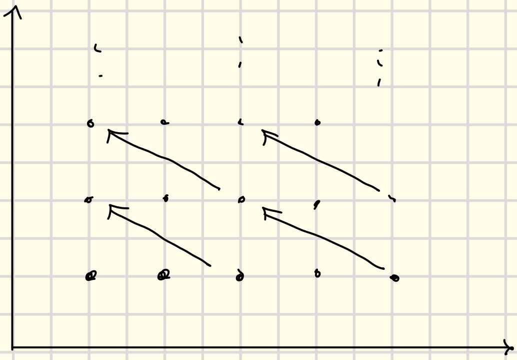
DEF: a COHOMOLOGY SPEC. SEQ. (CSS) is given by

- $\{E_r^{p,q}\}$ objects
- maps $d_r^{pq} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$
- iso. with cohomology

E^1_{**} :



$E^2_{*,*}$:



We say that an HSS is BOUNDED if $\forall m$ there are only finitely many nonzero term of total degree m in $E^a_{*,*}$

↳ $m = p+q$

If the HSS is bounded, then for each p,q $\exists r_0$ s.t. $E^r_{p,q} = E^{r_0}_{p,q}$ $\forall r \geq r_0$

We will denote them with $E^\infty_{p,q}$

Def: a bounded HSS converges to H_* if we have a family of objects H_m with finite filtrations

$$0 = F_s H_m \subseteq \dots \subseteq F_{p-1} H_m \subseteq F_p H_m \subseteq \dots \subseteq F_t H_m = H_m$$

s.t. $E^\infty_{p,q} \cong F_p H_{p+q} / F_{p-1} H_{p+q}$

Notation: $E^a_{p,q} \Rightarrow H_{p+q}$

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STEP BACK:

DEF: a FILTRATION of a space X is a seq. of subspaces

$$\dots \subset X_0 \subset X_1 \subset \dots \subset X.$$

Recall: given a pair (X, A), with $A \subset X$, the RELATIVE HOMOLOGY is

$$H_n(X, A; G) := H_n(C_*(X)/C_*(A))$$

THM: we have a long exact sequence:

$$\longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \dots$$

Suppose we have a finite filtration $0 = X_0 \subset \dots \subset X$

STAIRCASE DIAGRAM:

$$\begin{tikzcd}
\arrow[d] & & & \\
H_{n+1}(X_p) \arrow[r] \arrow[d] & H_{n+1}(X_p, X_{p-1}) \arrow[r] & H_n(X_{p-1}) \arrow[r] \arrow[d] & {} \\
H_{n+1}(X_{p+1}) \arrow[r] & H_{n+1}(X_{p+1}, X_p) \arrow[r] & H_n(X_p) \arrow[r] \arrow[d] & H_n(X_p, X_{p-1}) \arrow[r] & \dots \\
& & \vdots &
\end{tikzcd}$$

DEF: an EXACT COUPLE is a pair of bigraded abelian groups A and E along with bigraded maps i, j, k s.t.:

$$\begin{tikzcd}
A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\
& E \arrow[ul, "k"] &
\end{tikzcd}$$

is exact

$\operatorname{Im} j = \operatorname{Ker} k$

$\operatorname{Im} k = \operatorname{Ker} i$

$\operatorname{Im} i = \operatorname{Ker} j$

DEF: a DERIVED COUPLE is an exact couple associated to the exact couple (A, E, i, j, k) with

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

s.t. :
- $A' = i(A)$
- $E' = \mathrm{Ker}(jk) / \mathrm{Im}(jk)$ $\qquad jk = d$
- $i' = i|_{i(A)}$
- $\forall\, i(a) \in A'$, $j'(i(a)) = [j(a)] \in E'$
- $\forall\, [e] \in E'$, $k'([e]) = k(e)$

RMK : for every exact couple $d := jk$ is s.t. $d^2 = 0$

↳ $jkjk = j\underbrace{(kj)}_{=0}k = 0$

LEMMA: from a filtration we can construct an exact couple

↳ $\forall\, p,q \in \mathbb{Z}$ let $E^1_{p,q} := H_{p+q}(X_p, X_{p-1})$
$A^1_{p,q} := H_{p+q}(X_p)$ $\Rightarrow$ $E^1 = \bigoplus_{p,q} E^1_{p,q}$, $A^1 = \bigoplus_{p,q} A^1_{p,q}$

$i^1 : A^1 \longrightarrow A^1$ is induced by the inclusions $X_{p-1} \hookrightarrow X_p$

$j^1 : A^1 \longrightarrow E^1$ is induced by $H_*(X_p) \longrightarrow H_*(X_p, X_{p-1})$

$k^1 : E^1 \longrightarrow A^1$ is given by the boundary maps

RMK: i is $(1,-1)$ graded

j is $(0,0)$ "

k is $(-1,0)$ "

LERAY - SERRE SPEC. SEQ.

Suppose $\pi : X \longrightarrow B$ is a fibration with B a path-conn. CW-complex

DEF: we say that π has the homotopy lifting property (HLP) w. r. Y

if

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "h_0"] \arrow[d, hook] & X \arrow[d, "\pi"] \\ Y \times [0,1] \arrow[r, "h"'] \arrow[ur, dashed, "\exists \tilde{h}"] & B \end{tikzcd}$$

π is a fibration if it has the HLP w. r. to every Y

We can consider the filtration $X_p := \pi^{-1}(B^p)$, B^p = p - skeleton

We explicitly construct a HSS with :

$$E^1_{p,q} := H_{p+q}(X_p, X_{p-1})$$

THM (Serre): let $F \longrightarrow X \xrightarrow{\pi} B$ be a fibration with B a simply-conn. CW-complex

Then there is a HSS s.t. :

- $E^2_{p,q} \cong H_p(B ; H_q(F;G))$
- $E^\infty_{p,m-p} \cong F^p_m / F^{p-1}_m$ with $0 \subset F^0_m \subset \ldots \subset F^m_m = H_m(X;G)$

$\mathbb{CP}^\infty$: $S^1 \longrightarrow S^\infty \longrightarrow \mathbb{CP}^\infty$

We can apply the thm. : $E^2_{p,q} = H_p(B ; H_q(F;\mathbb{Z}))$

$$H_q(F;\mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0, 1 \\ 0 & \text{else} \end{cases}$$

$\Rightarrow$ $E^2_{p,q} = 0$ for $q \neq 0, 1$ and $p < 0$

II page:

$$\begin{array}{c|cccc} 1 & 0 & E^2_{0,1} & E^2_{1,1} & \cdots \\ 0 & 0 & E^2_{0,0} & E^2_{1,0} & \cdots \\ -1 & 0 & 0 & 0 & \cdots \\ \hline & -1 & 0 & 1 & \end{array}$$

$$\wr$$

$$\begin{tikzcd} 0 & H_0(\mathbb{CP}^\infty) \arrow[l] & H_1(\mathbb{CP}^\infty) \arrow[d, "\sim"] & H_2(\mathbb{CP}^\infty) & \cdots & \\ & \mathbb{Z} \cong H_0(\mathbb{CP}^\infty) & H_1(\mathbb{CP}^\infty) & H_2(\mathbb{CP}^\infty) \arrow[llu] & \cdots \arrow[llu] & 0 \arrow[ll] \end{tikzcd}$$

RMK: • $E^2_{p,q} = 0 \implies E^\infty_{p,q} = 0$

• $E^\infty_{p,m-p} \cong F^p_m / F^{p-1}_m$ with $0 \subset F^0_m \subset \dots \subset H_m(X;\mathbb{Z})$

But $H_m(S^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z} & m=0 \\ 0 & \text{else} \end{cases}$

$\Downarrow$

$E^\infty_{p,m-p} = 0 \quad \forall\, m \neq 0$

$E^\infty_{0,0} = \mathbb{Z}$

$$E^\infty : \quad \begin{array}{cccc} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \mathbb{Z} & 0 & 0 & \cdots \end{array}$$

LEMMA: $E^2_{0,0} = E^\infty_{0,0} = \mathbb{Z}$

$E^3_{p,q} = E^\infty_{p,q}$

$$H_m(\mathbb{CP}^\infty) = \begin{cases} \mathbb{Z} & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$$