Group Homology and Cohomology

- Definitions
- Example of $\mathbb{Z}$
- Low (Co)homology groups
- Shapiro Lemma.

§1 Let G be a group.

<u>Defn</u>: A (left) G-module is an abelian group A on which G acts by additive maps on the left.
i.e. $G \times A \longrightarrow A$

i) $g \cdot (a+b) = g \cdot a + g \cdot b$

ii) $1_G \cdot a = a$

iii) $gh \cdot a = g \cdot (h \cdot a)$

Let $\mathrm{Hom}_G(A, B)$ denote G-equivariant maps between G-modules A to B.

<u>Rem</u> G-mod $\cong$ $\mathbb{Z}G$-mod the category of (ring) modules over the ring $\mathbb{Z}G$.

<u>Recall</u> $\mathbb{Z}G$ are formal linear combinations of elements in G with $\mathbb{Z}$ coefficients.

$$\mathbb{Z}G = \left\{ \sum_{g \in G} n_g g \; : \; n_g \in \mathbb{Z} \text{ and non-zero for finitely many } g \right\}$$

- $\displaystyle\sum_{g \in G} n_g g + \sum_{g \in G} m_g g = \sum_{g \in G} (n_g + m_g) g$
- $\displaystyle\sum_{g \in G} n_g g \cdot \sum_{h \in G} m_h h = \sum_{g, h \in G} n_g m_h gh$

<u>Rem</u> $G\text{-mod} \cong Ab^G$ where G is 1-object category.
Implies G-mod is abelian category.

Given a G-module A denote <u>invariants</u> as

$$A^G = \{ a \in A : g \cdot a = a \;\; \forall g \in G \}$$

And <u>Coinvariants</u>

$$A_G = A / \langle (ga - a) \mid g \in G \;\; a \in A \rangle$$

Submodule generated by...

$(-)^G$ and $(-)_G$ are functors $G\text{-mod} \to Ab$.

Lemma/Exercise: Let $\mathrm{Triv}: Ab \to G\text{-mod}$ be the functor taking an abelian group to itself with trivial G-action. Then,

$(\_)_G$ is left-adjoint to triv and

$(\_)^G$ is right-adjoint to Triv.

Recall adjoints: F left adjoint

$$\begin{tikzcd}
F(Y) \arrow[d, dashed, "\exists! F(g)"'] \arrow[dr, "\forall f"] & \\
F(G(X)) \arrow[r, "\varepsilon_X"'] & X
\end{tikzcd}$$

G right adjoint

$$\begin{tikzcd}
Y \arrow[r, "\eta_Y"] \arrow[dr, "\forall g"'] & G(F(Y)) \arrow[d, dashed, "\exists! G(f)"] \\
& G(X)
\end{tikzcd}$$

$A \in G\text{-mod}$, $B \in Ab$

$$\begin{tikzcd}
A_G \arrow[d, dashed, "F(g)"'] \arrow[dr, "f"] & \\
B_G \arrow[r, "\cong"] & B
\end{tikzcd}$$

$g: A \xrightarrow{q} A_G \xrightarrow{f} B_G$

is in G-mod as "Bad notation!"

$f(g \cdot a) = f(a) = g \cdot f(a)$

$A \in G\text{-mod}$, $B \in Ab$

$$\begin{tikzcd}
B \arrow[r, "\cong"] \arrow[dr, "g"'] & B \arrow[d, dashed, "g"] \\
& A_G
\end{tikzcd}$$

g maps invariants to invariants

· Now we can define H_* and H^*

Recall: Left adjoints $\Rightarrow$ Left derived functor
Right adjoint $\Rightarrow$ Right derived functor

Defn
$$H_*(G, A) := L_*(-_G)(A)$$
$$H^*(G, A) := R_*(-^G)(A)$$

Lem. Let A be G-module, $\mathbb{Z}$ a trivial G-module. Then, $A_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} A$ and $A^G \cong \mathrm{Hom}_G(\mathbb{Z}, A)$.

Proof of second identity: $A^G \cong \mathrm{Hom}_{Ab}(\mathbb{Z}, A^G)$
"By adjunction" $\cong \mathrm{Hom}_G(\mathbb{Z}, A)$
$\cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$

Recall left-derived functor of $\mathbb{Z} \otimes_{\mathbb{Z}G} -$ is $\mathrm{Tor}^{\mathbb{Z}G}_*(\mathbb{Z}, -)$

And right-derived functor of $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -)$ is $\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, -)$.

First example $G = \langle t \rangle$.

Note $\mathbb{Z}G \cong \mathbb{Z}[t, t^{-1}]$

and we have exact sequence

$$* \quad \underbrace{0 \to \mathbb{Z}G \xrightarrow{(t-1)\cdot} \mathbb{Z}G}\xrightarrow{ev_1} \mathbb{Z} \to 0$$

This is projective resolution of $\mathbb{Z}$.

Apply $- \otimes_{\mathbb{Z}G} A$: $0 \longrightarrow A \xrightarrow{(t-1)\cdot} A \longrightarrow 0$

So, $H_0(G; A) = A / \mathrm{Im}(t-1) = A / (t-1)\cdot A \cong A_G$

And, $H_1(G, A) = \mathrm{Ker}(t-1) = \{a \in A \mid g \cdot a = a \ \forall g \in G\}$

$= A^G$

- $H_n(G, A) = 0$ for $n > 1$

For Cohomology.

Take $*$ and apply $\mathrm{Hom}_{\mathbb{Z}G}(-, A)$ then

$$0 \leftarrow \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \xleftarrow{(t-1)} \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \to 0$$

$$0 \leftarrow A \xleftarrow{(t-1)\cdot} A \leftarrow 0$$

Our chain complex is reverse of before so

$H^0(G, A) = A^G \quad H_1(G, A) = A_G \quad H_n(G, A) = 0$

$\forall n > 1.$

In previous example we had map
$ev_1 : \mathbb{Z}G \to \mathbb{Z}$ defined $\sum_{g\in G} n_g g \mapsto \sum_{g \in G} n_g$

- The kernel of ev_1 is the augmentation ideal of $\mathbb{Z}G$. Denoted I

FACT 1) I is a free $\mathbb{Z}$-module with basis $\{ g-1 : g \in G, g \neq 1 \}$.

Fact 2) $I/I^2 \cong G/[G,G]$

<u>Thm</u> For any group G, $H_1(G;\mathbb{Z}) \cong G/[G,G]$

<u>Proof</u> We have s.e.s.

$$0 \to I \to \mathbb{Z}G \xrightarrow{ev_1} \mathbb{Z} \to 0$$

By property of derived functors induces l.e.s.

$$H_1(G,\mathbb{Z}G) \to H_1(G,\mathbb{Z}) \to I_G \to (\mathbb{Z}G)_G \to \mathbb{Z} \to 0$$

- Since $\mathbb{Z}G$ is projective $H_1(G,\mathbb{Z}G) = 0$.

Note $g \cdot a - a = 0 = (g-1)\cdot a$ so

$$(\mathbb{Z}G)_G = \mathbb{Z}G/(g-1)\mathbb{Z}G \cong \mathbb{Z}G/I \cong \mathbb{Z} \text{ by s.e.s.}$$

Therefore our Sequence becomes

$$0 \to H_1(G, \mathbb{Z}) \to I_G \to 0$$

What is I_G? $\mathcal{I}/_{(g-1)\mathcal{I}} \cong \mathcal{I}/_{\mathcal{I}^2} \cong G/_{[G,G]}$.

for all $g \in \mathcal{I}$

$\square$

What about $H^1(G, \mathbb{Z})$?

· Last week we saw UCT theorem

$$0 \to Ext^1_R(H_{n-1}(P_\cdot), M) \to H^n(Hom_R(P_\cdot, M)) \to Hom_R(H_n(P_\cdot), M) \to 0$$

where $P_\cdot$ is Chain complex of projectives.

. For group cohomology let A be trivial G-module

$$0 \to Ext^1_{\mathbb{Z}}(H_{n-1}(G, \mathbb{Z}), A) \to H^n(G, A) \to Hom_{Ab}(H_n(G, \mathbb{Z}), A) \to 0$$

· We showed $H_0(G, \mathbb{Z}) = (\mathbb{Z}G)_G \cong \mathbb{Z}$

so $Ext^1_{\mathbb{Z}}(H_0(G, \mathbb{Z}), \mathbb{Z}) = Ext^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0$

· Therefore $H^1(G, \mathbb{Z}) \cong Hom_{Ab}(H_1(G, \mathbb{Z}), \mathbb{Z})$

- Since $H_1(G, \mathbb{Z}) = G^{ab}$ abelinazation

~~G~~ $H^1(G, \mathbb{Z}) \cong Hom_{Ab}(G^{ab}, \mathbb{Z})$

- Moreover is G is finite $\Rightarrow G^{ab}$ finite

so $Hom_{Ab}(G^{ab}, \mathbb{Z}) = 0$.

Shapiro Lemma

Background: $R \subseteq S$ rings then for right R-module M and right S-module N

- $M \otimes_R S$ is induced S-module
- $Hom_R(S, M)$ is Coinduced S-module
- N_R is restriction of scalars

<u>Lemma</u>
- Induced module left adjoint to restriction
- Coinduced is right adjoint to restriction

<u>Defn</u> Let $H \leq G$ and A a left $\mathbb{Z}H$-module then

- $Ind_H^G(A) = \mathbb{Z}G \otimes_{\mathbb{Z}H} A$ "Induced"
- $Coind_H^G(A) = Hom_H(\mathbb{Z}G, A)$

Shapiro Lemma: Let $H \leq G$ and A a H-module then

$$H_*(G, \mathrm{Ind}_H^G(A)) \cong H_*(H, A) \quad \text{and}$$

$$H^*(G, \mathrm{Coind}_H^G(A)) \cong H^*(H, A).$$

Proof: • $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module (Coset representatives form basis).

• Therefore if P is projective $\mathbb{Z}G$-module we have

$$P \otimes_{\mathbb{Z}G} \underbrace{(\mathbb{Z}G \otimes_{\mathbb{Z}H} A)}_{\mathrm{Ind}_H^G(A)} \cong P \otimes_{\mathbb{Z}H} A$$

. We calculate $H_*(G, \mathrm{Ind}_H^G(A))$

. Take $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z}$ proj. resolution ~~of~~ in $\mathbb{Z}G$. This is proj res in $\mathbb{Z}H$.

. Tensor with $\mathrm{Ind}_H^G(A)$

$$\cdots \longrightarrow P_1 \otimes_{\mathbb{Z}H} A \longrightarrow P_0 \otimes_{\mathbb{Z}H} A \longrightarrow 0$$

. Take homology: $H_n(G, \mathrm{Ind}_H^G(A)) \cong H_n(H, A)$.

For Cohomology $H^*(G, \mathrm{Coind}_H^G(A))$ consider same proj resolution then

$$\mathrm{Hom}_G(P_n, \mathrm{Hom}_H(\mathbb{Z}G, A)) \cong \mathrm{Hom}_H(P, A)$$

by adjunction of Coinduced and restriction.

So $H^*(G, \mathrm{Coind}_H^G(A)) \cong H^*(H, A)$ $\square$

Trivial group $\{1\}$. Note

$$H_*(\{1\}, A) = H^*(\{1\}, A) = \begin{cases} A & * = 0 \\ 0 & * \neq 0 \end{cases}$$

So by Shapiro Lemma

$$H_*(G, \mathbb{Z}G \otimes_{\mathbb{Z}} A) = H^*(G, \mathrm{Hom}_{Ab}(\mathbb{Z}G, A)) = \begin{cases} A & * = 0 \\ 0 & * \neq 0 \end{cases}$$

<u>Lem</u> If $[G:H]$ is finite, $\mathrm{Ind}_H^G(A) \cong \mathrm{Coind}_H^G(A)$.

So for G finite $[G:\{1\}]$ is finite so

$$\mathrm{Coind}_{\{1\}}^G(\mathbb{Z}) = \mathrm{Ind}_{\{1\}}^G(\mathbb{Z}) = \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}G$$

$$H_*(G, \mathbb{Z}G) = H^*(G, \mathbb{Z}G) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * \neq 0 \end{cases}$$

Rem · We proved $\cancel{H^0(\mathbb{Z}, \mathbb{Z}[G]) \cong \mathbb{Z}[\mathbb{Z}]^{\mathbb{Z}}}$

If $G = \mathbb{Z}$ then

$$H^1(G, \mathbb{Z}G) \cong \mathbb{Z}G_G \cong \mathbb{Z}$$

So $H^1(G, \mathbb{Z}G)$ can distinguish some infinite and finite groups.