Homological algebra — Week 5 (Tor and Ext once more! And homological dimension)

R ring (non-commutative), $R\text{-Mod}$ (left), $\text{Mod-}R$ (right)

Tor: (left) derived functor of $\otimes_R : \text{Mod-}R \times R\text{-Mod} \longrightarrow Ab$ $\quad A \otimes_R B$ ↗ right exact

$-\otimes_R B : \text{Mod-}R \to Ab$	$A \otimes_R - : R\text{-Mod} \to Ab$
1) Projective resolution of A	1) Projective res. of B
$\cdots \to P_1 \to P_0 \to A \to 0$	$\cdots \to Q_1 \to Q_0 \to B \to 0$
2) Apply $\otimes$: $\cdots \to P_1 \otimes_R B \to P_0 \otimes_R B \to 0$	2) Apply $\otimes$: $\cdots \to A \otimes_R Q_1 \to A \otimes_R Q_0 \to 0$
3) $\mathrm{Tor}_n^R(A,B)$ is the homology of this.	3) $\mathrm{Tor}_n^R(A,B)$ is the homology

Non-trivial fact: these two computations agree.

Facts: $\mathrm{Tor}_*(A, \bigoplus_{i\in I} B_i) = \bigoplus_{i \in I} \mathrm{Tor}_*(A, B_i)$ (in both arguments)

- If R is commutative, $\mathrm{Tor}_i^R(A,B) \cong \mathrm{Tor}_i^R(B,A)$.

Ext: (right) derived functor of $\mathrm{Hom}_R : R\text{-Mod}^{op} \times R\text{-Mod} \longrightarrow Ab$ (left exact)

$\mathrm{Hom}_R(A,-) : R\text{-Mod} \to Ab$	$\mathrm{Hom}_R(-,B) : R\text{-Mod}^{op} \to Ab$
• Injective resolution of B	• Projective resolution of A
$0 \to B \to I^0 \to I^1 \to \cdots$	$\cdots$
$0 \to \mathrm{Hom}_R(A,B) \to \mathrm{Hom}_R(A, I^0) \to \cdots$	
$\mathrm{Ext}_R^n(A,B)$	$\mathrm{Ext}_R^n(A,B)$

Again: these are the same (non-trivial).

Fact: $\mathrm{Ext}_R^n(\bigoplus_\alpha A_\alpha, B) \cong \prod_\alpha \mathrm{Ext}_R^n(A_\alpha, B)$ (of course, finite sums = products)

$\mathrm{Ext}_R^n(A, \prod_\beta B_\beta) = \prod_\beta \mathrm{Ext}_R^n(A, B_\beta)$

- Why is Tor? (torsion)

→ Abelian groups ($R = \mathbb{Z}$) $\qquad$ Question 1: $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/p, B) = ?$ (p not necessarily prime)

$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0$ $\quad$ projective res. of $\mathbb{Z}/p$ in Ab

$\otimes B$

$0 \to \underset{1}{B} \xrightarrow{p} \underset{0}{B} \to 0$ $\qquad$ $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/p, B) = 0$ for $n \geq 2$

$\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/p, B) = B/pB = \mathbb{Z}/p \otimes B$

$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p, B) = {}_pB = \{b \in B : pb = 0\}$ "p-torsion" $\qquad$ $pB = \{pb : b \in B\}$ $\neq {}_pB$

→ A = finitely generated abelian group? $A \cong \mathbb{Z}^m \times \mathbb{Z}/p_1 \times \cdots \times \mathbb{Z}/p_n$

$\mathrm{Tor}_i^{\mathbb{Z}}(A,B) = \cancel{\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}^m, B)} \oplus \mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/p_1, B) \oplus \cdots \oplus \mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/p_n, B)$

$=0$ projective

E.g. $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/3, \mathbb{Z}/7) = 0$ $\qquad$ $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/q) = \begin{cases} \mathbb{Z}/p & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases}$

$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/5, \mathbb{Z}/5) = \mathbb{Z}/5$ $\qquad$ p, q prime

$\mathrm{Tor}_1^{\mathbb{Z}}(\underset{\mathbb{Z}/2 \times \mathbb{Z}/5}{\mathbb{Z}/10}, \underset{\mathbb{Z}/3 \times \mathbb{Z}_5}{\mathbb{Z}/15}) = \mathbb{Z}/5$, in general $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/m) = \mathbb{Z}/\gcd(m,n)$

$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/100, \mathbb{Z}/1000) = \mathbb{Z}/100$ $\qquad$ • $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B)$ = torsion subgroup of B

In general: for any $A, B \in Ab$, $\boxed{\mathrm{Tor}_n^{\mathbb{Z}}(A,B) = 0, \ n \geq 2}$

(A is the direct limit of its finitely generated subgroups)

Other example: $R = \mathbb{Z}/8$, $A = \mathbb{Z}/2$ $\qquad$ $\mathrm{Tor}_n^{\mathbb{Z}/8}(\mathbb{Z}/2, B) = ?$

projective resolution of $\mathbb{Z}/2$ in R-Mod:

$$\begin{tikzcd} & & & 0 \arrow[dr] & & 0 & & \\ & & & & \mathbb{Z}/2 \arrow[ur] \arrow[dr, "\times 4"] & & & \\ \cdots \arrow[r] & \mathbb{Z}/8 \arrow[r, "\times 4"] & \mathbb{Z}/8 \arrow[r, "\times 2"] & \mathbb{Z}/8 \arrow[r, "\times 4"] \arrow[ur] & \mathbb{Z}/8 \arrow[r, "\times 2"] \arrow[dr] & \mathbb{Z}/8 \arrow[r, "\text{mod } 2"] & \mathbb{Z}/2 \arrow[r] & 0 \\ & & & & & \mathbb{Z}/4 \arrow[u, "\times 2"] \arrow[dr] & & \\ & & & & 0 \arrow[ur] & & 0 & \end{tikzcd}$$

$\downarrow \otimes_R B$

$\cdots \to B \xrightarrow{\times 2} B \xrightarrow{\times 4} B \xrightarrow{\times 2} B \to 0$

$\mathrm{Tor}_n^{\mathbb{Z}/8}(\mathbb{Z}/2, B) = \begin{cases} B/2B, & n = 0 \\ \{b \in B : 2b = 0\}/4B, & n > 0 \text{ odd} \\ \{b \in B : 4b = 0\}/2B, & n > 0 \text{ even} \end{cases}$ $\qquad$ $B = \mathbb{Z}/4$?

$\mathrm{Tor}_n^{\mathbb{Z}/8}(\mathbb{Z}/2, \mathbb{Z}/4) \cong \begin{cases} \mathbb{Z}/2, & n = 0 \\ \mathbb{Z}/2, & n > 0 \text{ odd} \\ \mathbb{Z}/2, & n > 0 \text{ even} \end{cases}$

Corollary: $\mathbb{Z}/2$ and $\mathbb{Z}/4$ are not flat $\mathbb{Z}/8$-modules! (in particular, they are also not projectives)

- Tor can also be computed with flat resolutions (flat modules are $\otimes$-acyclic).

What about Ext?

- An extension of A by B is an exact sequence $0 \to B \to X \to A \to 0$.

Two extensions are equivalent if

$$\begin{tikzcd} & & X \arrow[dd, "\cong"] \arrow[dr] & & \\ 0 \arrow[r] & B \arrow[ur] \arrow[dr] & & A \arrow[r] & 0 \\ & & X' \arrow[ur] & & \end{tikzcd}$$

- An extension is split if it is equivalent to $0 \to B \to A \oplus B \to A \to 0$.

Lemma: If $\mathrm{Ext}^1(A,B) = 0$, then every extension of A by B is split.

$0 \to B \to X \to A \to 0$ $\quad$ $\downarrow \mathrm{Hom}(A,-)$

$0 \to \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,X) \to \mathrm{Hom}(A,A) \xrightarrow{\partial} \underset{0}{\mathrm{Ext}^1(A,B)} \to \cdots$

surjective $\quad$ id_A

$$\begin{tikzcd} & & & A \arrow[dl, dashed] \arrow[d, "id_A"] & \\ 0 \arrow[r] & B \arrow[r] & X \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

the sequence splits

↳ More precisely: the obstruction to an extension being split is the class $\partial\, id_A \in \mathrm{Ext}^1(A,B)$.

$0 \to B \to X \to A \to 0 \ \mapsto \ \partial\, id_A$

Theorem: $A, B \in R$-mod, there is a bijection $\left\{ \begin{matrix} \text{eq. classes of} \\ \text{extensions of } A \text{ by } B \end{matrix} \right\} \xleftrightarrow{1-1} \mathrm{Ext}^1(A,B)$.

There is an operation on the left side (Baer sum) which make it into an abelian group. and then the bijection is an iso. of ab. groups.

Example: p prime, what are the extensions of $\mathbb{Z}/p$ by $\mathbb{Z}/p$? $\mathrm{Ext}^1(\mathbb{Z}/p, \mathbb{Z}/p) = ?$

$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0$

$\downarrow \mathrm{Hom}(-, \mathbb{Z}/p)$ $\qquad$ $f \longmapsto g(x) = f(px)$

$0 \to \underset{1}{\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/p)} \longrightarrow \underset{0}{\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/p)} \longrightarrow 0$

$\qquad \cong \qquad\qquad\qquad \cong$

$\mathbb{Z}/p \xrightarrow{\times p} \mathbb{Z}/p$

$f(1) \qquad\qquad g(1) = f(p) = p f(1)$

$\mathrm{Ext}^1(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{Z}/p \rightsquigarrow$ there are p extensions!

- the split extension: $0 \to \mathbb{Z}/p \to \mathbb{Z}/p \oplus \mathbb{Z}/p \to \mathbb{Z}/p \to 0$
- for each $i = 1, \ldots, p-1$: $0 \to \mathbb{Z}/p \xrightarrow{\times p} \mathbb{Z}/p^2 \xrightarrow{\times i} \mathbb{Z}/p \to 0$

Remark: There is a similar relationship between $\mathrm{Ext}^n(A,B)$ and $0 \to B \to X_n \to \cdots \to X_1 \to A \to 0$.

Universal coefficient theorem

Let $\cdots \to P_1 \to P_0 \to 0$ be a chain complex, we know the $H_n(P_\bullet)$.
Can we compute the homology of the complex $P \otimes M$, for some $M \in R$-Mod?

$\cdots \to P_n \otimes M \to P_0 \otimes M \to 0$

$H_n(P \otimes M) \cong H_n(P) \otimes M$? No...

Theorem: [Künneth formula]
Let $P.$ be a chain complex of flat right R-modules s.t. each submodule $d(P_n) \subseteq P_{n-1}$ is also flat. Then, exact sequence

$$0 \to H_n(P.) \otimes_R M \to H_n(P. \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(P.), M) \to 0.$$

If $R = \mathbb{Z}$ and $P.$ are free abelian groups. then the sequence splits (non-canonically).

Consequence in topology: X topological space. $H_n(X)$ singular homology of a complex $C.(X)$

$H_n(X; M) \cong (H_n(X) \otimes M) \oplus \mathrm{Tor}_1(H_{n-1}(X), M)$.

Example: $X = \mathbb{RP}^2$ $\qquad H_0 = \mathbb{Z}$. $H_1 = \mathbb{Z}/2$. $H_2 = 0$ $\quad H_n = 0$ $n \geq 3$

↳ $H_n(\mathbb{RP}^2; \mathbb{Z}/2) = ?$

Version for cohomology: ~~flat~~ projective

$$0 \to \mathrm{Ext}_R^1(H_{n-1}(P), M) \to H^n(\mathrm{Hom}_R(P, M)) \to \mathrm{Hom}_R(H_n(P), M) \to 0.$$

Examples: $H^n(\mathbb{RP}^2, \mathbb{Z}) = ?$ $\qquad H^n(\mathbb{RP}^2, \mathbb{Z}/2) = ?$

Also: homology of a product

$$0 \to \bigoplus_{p=0}^{n} H_p(X) \otimes H_{n-p}(Y) \to H_n(X \times Y) \to \bigoplus_{p=1}^{n} \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(X), H_{n-p}(Y)) \to 0$$

Def: $A \in \text{Mod-}R$

- The projective dim. of A, $pd(A)$, is the length of the shortest projective resolution of A. (can be $= \infty$)

$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$ $\qquad pd(A) = n$

Similarly: injective dimension, $id(A)$ $\qquad$ projectives ⇒ flat

flat dimension, $fd(A)$ $\qquad$ thus $fd(A) \leq pd(A)$

Global dimension theorem: For any ring R, the following numbers are the same:

1) $\sup\{id(B) : B \in \text{Mod-}R\}$
2) $\sup\{pd(A) : A \in \text{Mod-}R\}$
3) $\sup\{pd(R/I) : I \text{ right ideal of } R\}$
4) $\sup\{d : \mathrm{Ext}_R^d(A,B) \neq 0 \text{ for some } A, B\}$.

This number is the (right) global dimension of R, $r.gl.dim(R)$.

Remark: In general, $r.gl.dim(R) \neq l.gl.dim(R)$.

- They agree if R is commutative.
- They also agree if R is both left and right Noetherian.

Tor-dimension: (weak global dimension)

$\sup\{fd(A) : A \in \text{Mod-}R\}$
$\sup\{fd(B), B \in R\text{-Mod}\}$
$\sup\{d : \mathrm{Tor}_d(A,B) \neq 0\}$

$\text{Tor-dim}(R) \leq r.gl.dim.(R)$

- $R = k$ a field, then $gl.dim(k) = 0$
- $R = \mathbb{Z}$: $gl.dim(\mathbb{Z}) = \text{Tor dim}(\mathbb{Z}) = 1$
- Example: $R = \mathbb{Z}/8$, $gl.dim(\mathbb{Z}/8) = \text{Tor-dim}(\mathbb{Z}/8) = \infty$
- $\text{Tor-dim}(\prod_{i=0}^{\infty} \mathbb{C}) = 0$, $\quad gl.dim(\prod_{i=0}^{\infty} \mathbb{C}) \geq 2$

Fact: $gl.dim(\prod_{i=1}^{\infty} \mathbb{C}) = 2 \iff$ the continuum hypothesis holds.