

Group Homology and Cohomology

- Definitions
- Example of $\mathbb{Z}$
- Low (Co)homology groups
- Shapiro Lemma.

§1 Let G be a group.

Defn: A (left) G -module is an abelian group A on which G acts by additive maps on the left.

i.e. $G \times A \rightarrow A$

i) $g \cdot (a+b) = g \cdot a + g \cdot b$

ii) $1_G \cdot a = a$

iii) $gh \cdot a = g \cdot (h \cdot a)$

Let $\text{Hom}_G(A, B)$ denote G -equivariant maps between G -modules A to B .

Rem $G\text{-mod} \cong \mathbb{Z}G\text{-mod}$ the category of (ring) modules over the ring $\mathbb{Z}G$.

Recall $\mathbb{Z}G$ are formal linear combinations of elements in G with $\mathbb{Z}$ coefficients.

$$\mathbb{Z}G = \left\{ \sum_{g \in G} n_g g : n_g \in \mathbb{Z} \text{ and non-zero for finitely many } g \right\}$$

$$\cdot \quad \sum_{g \in G} n_g g + \sum_{g \in G} m_g g = \sum_{g \in G} (n_g + m_g) g$$

$$\cdot \quad \sum_{g \in G} n_g g \cdot \sum_{h \in G} m_h h = \sum_{g, h \in G} n_g m_h gh$$

Rem $G\text{-mod} \cong \text{Ab}^G$ where G is 1-object category.
 Implies $G\text{-mod}$ is abelian category.

Given a G -module A denote invariants as
 $A^G = \{ a \in A : g \cdot a = a \ \forall g \in G \}$

And Coinvariants

$$A_G = A / \langle (ga - a) \mid g \in G, a \in A \rangle$$

Submodule generated by...

$(-)^G$ and $(-)_G$ are functors $G\text{-mod} \rightarrow \text{Ab}$.

Lemma/Exercise: Let $\text{Triv}: \text{Ab} \rightarrow G\text{-mod}$ be the functor taking an abelian group to itself with trivial G -action. Then,

$(-)_G$ is left-adjoint to Triv and
 $(-)^G$ is right-adjoint to Triv .

Recall adjoints: F left adjoint

$$\begin{array}{ccc} F(Y) & & \\ \downarrow \exists! F(g) & \searrow \forall g & \\ F(G(X)) & \xrightarrow{\epsilon_X} & X \end{array}$$

$A \in G\text{-mod}, B \in \text{Ab}$

$$\begin{array}{ccc} A_G & & \\ \downarrow F(g) & \searrow g & \\ B_G & \xrightarrow{\cong} & B \end{array}$$

$$g: A \xrightarrow{q} A_G \xrightarrow{g} B_G$$

is in $G\text{-mod}$ as "Bad notation!"

$$g(g \cdot a) = g(a) = g \cdot g(a)$$

G right adjoint

$$\begin{array}{ccc} Y & \xrightarrow{m_Y} & G(F(Y)) \\ & \searrow \forall g & \downarrow \exists! G(g) \\ & & G(X) \end{array}$$

$A \in G\text{-mod}, B \in \text{Ab}$

$$\begin{array}{ccc} B & \xrightarrow{\cong} & B \\ & \searrow g & \downarrow g \\ & & A_G \end{array}$$

g maps invariants to invariants

• Now we can define H_* and H^*

Recall: Left adjoints $\Rightarrow$ Left derived functor

Right adjoint $\Rightarrow$ Right derived functor

Defn

$$H_*(G, A) := L_*(-^G)(A)$$

$$H^*(G, A) := R_*(-^G)(A)$$

Lem. Let A be G -module, $\mathbb{Z}$ a trivial G -module. Then, $A_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} A$ and $A^G \cong \text{Hom}_G(\mathbb{Z}, A)$.

Proof of second identity: $A^G \cong \text{Hom}_{\text{Ab}}(\mathbb{Z}, A^G)$
"By adjunction" $\cong \text{Hom}_G(\mathbb{Z}, A)$
 $\cong \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$

Recall left-derived functor of $\mathbb{Z} \otimes_{\mathbb{Z}G} -$ is $\text{Tor}_{\mathbb{Z}G}^*(\mathbb{Z}, -)$

And right-derived functor of $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -)$ is $\text{Ext}_{\mathbb{Z}G}^*(\mathbb{Z}, -)$.

First Example $G = \langle t \rangle$.

Note $\mathbb{Z}G \cong \mathbb{Z}[t, t^{-1}]$

and we have exact sequence

$$* \quad 0 \rightarrow \mathbb{Z}G \xrightarrow{(t-1)} \mathbb{Z}G \xrightarrow{ev_1} \mathbb{Z} \rightarrow 0$$

This is projective resolution of $\mathbb{Z}$.

$$\text{Apply } - \otimes_{\mathbb{Z}G} A : 0 \rightarrow A \xrightarrow{(t-1)} A \rightarrow 0$$

$$\text{So, } H_0(G, A) = \frac{A}{\text{Im}(t-1)} = \frac{A}{(t-1) \cdot A} \cong A_G$$

$$\text{And, } H_1(G, A) = \text{Ker}(t-1) = \{a \in A \mid g \cdot a = a \ \forall g \in G\} \\ = A^G$$

$$\bullet H_n(G, A) = 0 \quad \text{for } n > 1$$

For Cohomology.

Take $*$ and apply $\text{Hom}_{\mathbb{Z}G}(-, A)$ then

$$0 \leftarrow \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \xleftarrow{(t-1)} \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \rightarrow 0$$

$$0 \leftarrow A \xleftarrow{(t-1)} A \leftarrow 0$$

Our Chain Complex is reverse of before so

$$H^0(G, A) = A^G \quad H^1(G, A) = A_G \quad H^n(G, A) = 0$$

$\forall n > 1$

In previous example we had map

$$\text{ev}_1: \mathbb{Z}G \rightarrow \mathbb{Z} \quad \text{defined} \quad \sum_{g \in G} n_g g \mapsto \sum_{g \in G} n_g$$

The kernel of ev_1 is the augmentation ideal of $\mathbb{Z}G$. Denoted I

FACT 1) I is a free $\mathbb{Z}$ -module with basis $\{g-1: g \in G, g \neq 1\}$.

$$\text{Fact 2)} \quad I/I^2 \cong G/[G, G]$$

Thm For any group G , $H_1(G; \mathbb{Z}) \cong G/[G, G]$

Proof We have S.E.S.

$$0 \rightarrow I \rightarrow \mathbb{Z}G \xrightarrow{\text{ev}_1} \mathbb{Z} \rightarrow 0$$

By property of derived functors induces L.E.S.

$$H_1(G, \mathbb{Z}G) \rightarrow H_1(G, \mathbb{Z}) \rightarrow I_G \rightarrow (\mathbb{Z}G)_G \rightarrow \mathbb{Z} \rightarrow 0$$

• Since $\mathbb{Z}G$ is projective $H_1(G, \mathbb{Z}G) = 0$.

Note $g \cdot a - a = 0 = (g-1) \cdot a$ so

$$(\mathbb{Z}G)_G = \mathbb{Z}G / (g-1)\mathbb{Z}G \cong \mathbb{Z}G / I \cong \mathbb{Z} \text{ by S.E.S.}$$

Therefore our sequence becomes

$$0 \rightarrow H_1(G, \mathbb{Z}) \rightarrow I_G \rightarrow 0$$

What is I_G ? $\mathbb{Z} / (g-1)\mathbb{Z} \cong \mathbb{Z} / \mathbb{Z}^2 \cong \mathbb{Z} / [G, G]$.

for all $g \in G$ □

What about $H^1(G, \mathbb{Z})$?

• Last week we saw UCT theorem

$$0 \rightarrow \text{Ext}_R^1(H_{n-1}(P), M) \rightarrow H^n(\text{Hom}_R(P, M)) \rightarrow \text{Hom}_R(H_n(P), M) \rightarrow 0$$

where P is chain complex of projectives.

• For group cohomology let A be trivial G -module

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(G, \mathbb{Z}), A) \rightarrow H^n(G, A) \rightarrow \text{Hom}_{\text{Ab}}(H_n(G, \mathbb{Z}), A) \rightarrow 0$$

• We showed $H_0(G, \mathbb{Z}) = (\mathbb{Z}G)_G \cong \mathbb{Z}$

so $\text{Ext}_{\mathbb{Z}}^1(H_0(G, \mathbb{Z}), \mathbb{Z}) = \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}, \mathbb{Z}) = 0$

• Therefore $H^1(G, \mathbb{Z}) \cong \text{Hom}_{\text{Ab}}(H_1(G, \mathbb{Z}), \mathbb{Z})$

• Since $H_1(G, \mathbb{Z}) = G^{ab}$ abelianization

by $H^1(G, \mathbb{Z}) \cong \text{Hom}_{Ab}(G^{ab}, \mathbb{Z})$

• Moreover if G is finite $\Rightarrow G^{ab}$ finite

so $\text{Hom}_{Ab}(G^{ab}, \mathbb{Z}) = 0$.

Shapiro Lemma

Background: $R \subseteq S$ rings then for
right R -module M and right S -module N

- $M \otimes_R S$ is induced S -module
- $\text{Hom}_R(S, M)$ is coinduced S -module
- N_R is restriction of scalars

Lemma • Induced module left adjoint to restriction
• Coinduced is right adjoint to restriction

Defn Let $H \leq G$ and A a left
 $\mathbb{Z}H$ -module then

• $\text{Ind}_H^G(A) = \mathbb{Z}G \otimes_{\mathbb{Z}H} A$ "Induced"

• $\text{Coind}_H^G(A) = \text{Hom}_H(\mathbb{Z}G, A)$

Shapiro Lemma: Let $H \trianglelefteq G$ and A a H -module then

$$H_*(G, \text{Ind}_H^G(A)) \cong H_*(H, A) \quad \text{and}$$

$$H^*(G, \text{Coind}_H^G(A)) \cong H^*(H, A).$$

Proof: • $\mathbb{Z}G$ is a free $\mathbb{Z}H$ -module
(Coset representatives form basis).

• Therefore if P is projective $\mathbb{Z}G$ -module we have

$$P \otimes_{\mathbb{Z}G} \underbrace{(\mathbb{Z}G \otimes_{\mathbb{Z}H} A)}_{\cong \text{Ind}_H^G(A)} \cong P \otimes_{\mathbb{Z}H} A$$

• We calculate $H_*(G, \text{Ind}_H^G(A))$

Take $\dots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z}$
proj. resolution of $\mathbb{Z}$ in $\mathbb{Z}G$. This is proj res in $\mathbb{Z}H$.

• Tensor with $\text{Ind}_H^G(A)$

$$\dots \rightarrow P_1 \otimes_{\mathbb{Z}H} A \rightarrow P_0 \otimes_{\mathbb{Z}H} A \rightarrow 0$$

• Take homology: $H_n(G, \text{Ind}_H^G(A)) \cong H_n(H, A).$

For Cohomology $H^*(G, \text{Coind}_H^G(A))$ Consider
 same proj resolution then

$$\text{Hom}_G(P_n, \text{Hom}_H(\mathbb{Z}G, A)) \cong \text{Hom}_H(P, A)$$

by adjunction of Coinduced and restriction.

So $H^*(G, \text{Coind}_H^G(A)) \cong H^*(H, A) \quad \square$

Trivial group $\{1\}$. Note

$$H_*(\{1\}, A) = H^*(\{1\}, A) = \begin{cases} A & * = 0 \\ 0 & * \neq 0 \end{cases}$$

So by Shapiro Lemma

$$H_*(G, \mathbb{Z}G \otimes_{\mathbb{Z}} A) = H^*(G, \text{Hom}_{\mathbb{A}G}(\mathbb{Z}G, A)) = \begin{cases} A & * = 0 \\ 0 & * \neq 0 \end{cases}$$

Lemma If $[G:H]$ is finite, $\text{Ind}_H^G(A) \cong \text{Coind}_H^G(A)$.

So for G finite $[G:\{1\}]$ is finite so

$$\text{Coind}_{\{1\}}^G(\mathbb{Z}) = \text{Ind}_{\{1\}}^G(\mathbb{Z}) = \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}G$$

$$H_*(G, \mathbb{Z}G) = H^*(G, \mathbb{Z}G) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * \neq 0 \end{cases}$$

Rem . We proved $H^0(\mathbb{Z}, \mathbb{Z}[g]) \cong \mathbb{Z}[g]$

1/8 $G = \mathbb{Z}$ then

$$H^1(G, \mathbb{Z}G) \cong \mathbb{Z}G \cong \mathbb{Z}$$

So $H^1(G, \mathbb{Z}G)$ can distinguish ^{some} finite

and finite groups.