Finite Type Invariants & Chord Diagrams.

Chmutov ch 3. & 4.

→ Goals:

* Define Vassiliev skein relations
* Define the Vassiliev knot invariant
* Introduce chord diagrams
* Show that all classical polynomials are Vassiliev
* Framed knots / Tangles

Origin : complements of discriminants? in spaces of maps.

Discriminants : subspaces of maps with singularities

$f: M \longrightarrow \mathbb{R}^3$, M 1 dimensional

$p \in im(f) \subset \mathbb{R}^3$, some point of our embedding M in $\mathbb{R}^3$.

→ $f^{-1}(p) \to t_1, t_2$

$f'(t_1)$ & $f'(t_2)$ linearly independent

Double point.

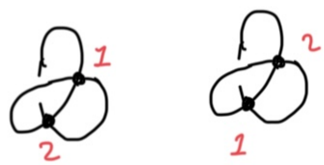
Vassiliev skein relation

$\nu($  $) = \nu($  $) - \nu($  $)$

"Resolving singularities of the knot"

* Notice $\nu(\ )$ is any classical knot invariant w/ values in some abelian group

* We can now extend the notion of classical invariants to knots with singularities.

→  $\rightsquigarrow$ $A\,\nu($  $) + B\,\nu($  $) + \ldots$

The result is *independent* of order.

Notice to calculate $\nu(K|_{n \text{ singular points}})$ reduce to all cases of "complete resolution".

$$\nu(K) = \sum_{\varepsilon_1 = \pm 1, \ldots \varepsilon_n = \pm 1} (-1)^{|\varepsilon|}\, \nu(K_{\varepsilon_1, \ldots \varepsilon_n})$$

$K \to$ singular knot

$K_{\varepsilon_1, \ldots \varepsilon_n} \to$ reduced knot (by some sequence of resolutions ε_i

Ex

Definition

A knot invariant is said to be a Vassiliev invariant of order $\leq n$ if its extension vanishes on all singular knots with more than n double points.

→ A Vassiliev invariant is order n if it is of order $\leq n$ but not $\leq n-1$.

For our purposes (some algebra.)

→ Vassiliev invariants take values in commutative rings.

$\mathcal{V}_n$ the set of Vassiliev invariants of order $\leq n$ with values in a ring R.

for each n, $\mathcal{V}_n$ is an R module.

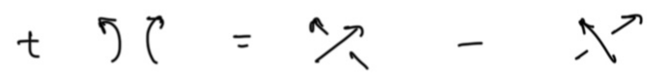
$$\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \cdots \subseteq \mathcal{V} := \bigcup_{n=0}^{\infty} \mathcal{V}_n$$

the set of Vassiliev invariants of order $\leq n$: $\mathcal{V}_n$

Example :

The n^{th} coefficient of the Conway polynomial is a Vassiliev invariant of order $\leq n$

Conway :



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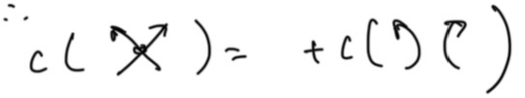

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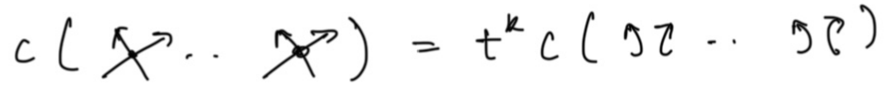
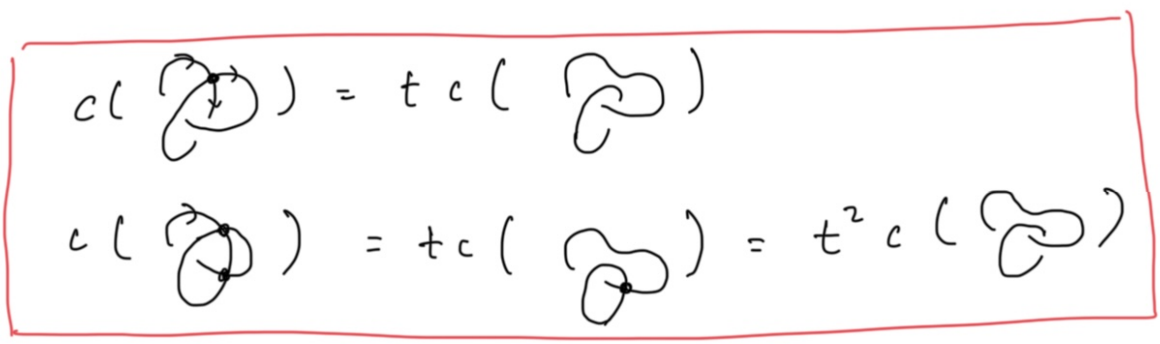




If $k \geq n+1$

$t^n = 0$

2 double points

$t^3 c(K) = 0$

Algebra of Vassiliev invariants

$K \to \mathbb{Z}K$ "Tautological knot invariant"

Knot with n singularities $\rightsquigarrow$
sent to alternating sum of 2^n genuine knots

$\mathbb{Z}K$: V space spanned by equivalence class
of knots.
Multiplication = connected sum. K_n is isotopy classes of n-singular

$K_n \rightsquigarrow \mathbb{Z}$ submodule of $\mathbb{Z}K$ spanned by
images of knots w/ n double points.

$K_n \to$ ideal of $\mathbb{Z}K$

Knot $n+1$ double points $\rightsquigarrow$ knot n double points in
$\mathbb{Z}K$.

K_0

$$\mathbb{Z}K = K_0 \supseteq K_1 \supseteq \cdots \supseteq K_n \supseteq \cdots$$

Definition

Let R be a commutative ring
A Vassiliev invariant of order $\leq n$ is a linear
function $\mathbb{Z}K \to R$ which vanishes on K_{n+1}

Vassiliev invariants as polynomials

$\delta \Rightarrow$ invariant on K_n

$\nabla(v) \Rightarrow$ Extension to K_{n+1}

$v : K \to \mathbb{R}$ is a Vassiliev invariant of degree $\leq n$ if

$\nabla^{n+1}(v) = 0$

approximations of continuous functions

: Polynomials

: Vassiliev invs

Invariants fail to be infinitely differentiable

Open Problem:

prove that any invariant can be approximated by Vassiliev type.

Approximating "classical" invariants:

$\boxed{\text{Polynomial \& Power Series Vassiliev invariants.}}$

$V_\bullet = \bigoplus_{n=0}^{\infty} V_n \rightarrow$ Polynomial Vassiliev invariant

Conway polynomial $\rightarrow$ <u>Power</u> series

* Rmk: A bit confusing as $c(K)$ is a polynomial

<u>Degrees 0, 1, 2</u>

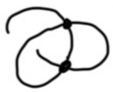
0.) $V_0 = \{\text{const}\}$, $\dim V_0 = 1$

Let <u>$f \in V_0$</u>

$K \rightarrow$ ⇝ change crossings $\rightarrow$ O

1.) $\nu_1 = \nu_0$

$\nu\Big($  $\Big)$ ⇝ can make a crossing change for "free"

$\nu\Big($  $\Big) = 0$

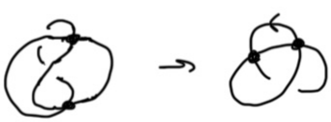
$\nu\Big($  $\Big) = \nu\Big($  $\Big) = 0$

2.) ν_2 (first non-trivial)

dim $\nu_2 = 2$

⇝ working w/ ν_2 can introduce 3rd singularity.

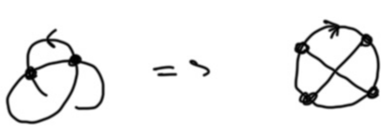
⇝ 



Chord diagrams

Definition:

A chord diagram of order n is an oriented circle with a distinguished set of n disjoint pairs of distinct points



Proposition The value of Vassiliev invariants of order $\leq n$ depends only on the chord diagram.

Hence there is a well defined map:

$\alpha_n : \mathcal{V}_n \to RA_n$

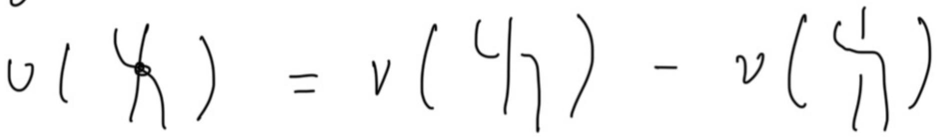
We would like to understand structure of $\mathcal{V}_n$:
$\rightsquigarrow$ $\ker(\alpha_n)$, $\operatorname{im}(\alpha_n)$

$\ker(\alpha_n) = \mathcal{V}_{n-1}$ (by defn.)

$\therefore \quad \bar{\alpha}_n : \mathcal{V}_n / \mathcal{V}_{n-1} \to RA_n$

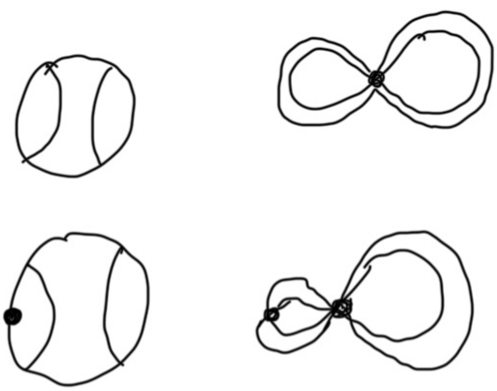
Framed Knots

we can extend the above to invariants of 'singular framed knots'.



Chord diagrams for framed knots

We can construct chord diagrams of framed knots
An equivalence class of framed singular knots



Classic polynomials via Vassiliev

Jones polynomial construction

→ Take Jones poly of a knot, let $t = e^h$

$j_n(k)$ is coeff of h^n

$\therefore$

→

Theorem

$j_n(k)$ is a Vassiliev invariant of order $\leq n$

$$t = e^h = 1 + h + \frac{h^2}{2} + \cdots$$

Jones

t^{-1}  $- \; t$  $= t^{1/2} - t^{-1/2}$ 

$\therefore$

$(1 - h + \cdots)$  $- \; (1 + h + \cdots)$ 

$= (h + \cdots)$ 

the diff between  $-$  $=$ 

can see that for a knot w/ $k > n+1$

singular points $h^n = 0$

$k = 1 \rightsquigarrow h^2 = 0$

 $\to$  $-$ 

J (trefoil) J (unknot)

$= e^h$ expansions of $= 1 + h$

Chord diagrams

Definition (four term relation)

for $f \in \mathbb{R}A_n$ where A_n are chord diagrams order n.

$f($  $) - f($  $)$

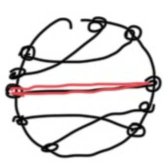
$+ f($  $) - f($  $) = 0$

where a function f satisfies the 4T relation it is called a <u>weight system</u> of order n.

Interpretation

$f($  $)$

Definition (Isolated chords)



A <u>unframed</u> weight system of order n is one in which the 1T relation holds

$\rightsquigarrow$ $f($ isol. $) = 0$

The fundamental theorem

Recall

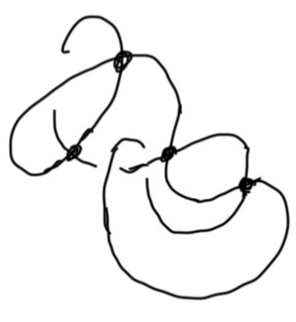
$$\bar{\alpha}_n : \mathcal{V}_n / \mathcal{V}_{n-1} \to \mathbb{R}A_n$$

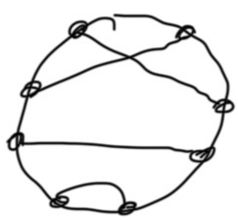
The space of unframed weight systems is isomorphic to the graded space associated to the filtered space of Vassiliev invariants

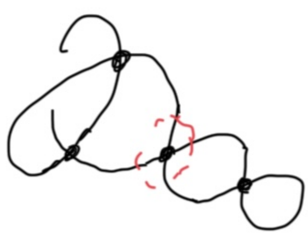
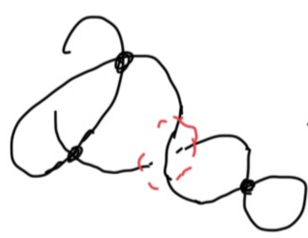
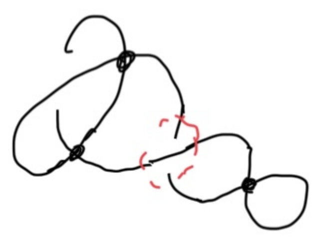
$$\mathcal{W} = \bigoplus_{n=0}^{\infty} \mathcal{W}_n \cong \bigoplus_{n=0}^{\infty} \mathcal{V}_n / \mathcal{V}_{n+1}$$

$\longrightarrow$ The symbol of every weight system is the symbol of a certain Vassiliev inv.

1T relations:





 $=$  $-$  $= 0$

framed knots

1T relation doesn't hold
and we have the 'framed' weight system
which only satisfies 4T
Proof of 4T will be done using Kontsevich.

Bialgebras

→ Some Vect space A over a field $\mathbb{F}$ s.t.

$\mu : A \otimes A \to A$ (product)

$\iota : \mathbb{F} \to A$ (unit)

$\delta : A \to A \otimes A$ (coproduct)

$\varepsilon : A \to \mathbb{F}$

with req/ commutativity

A BIALGEBRA has structure of algebra given μ, ι
coalgebra given δ, ε

s.t.

(1) $\varepsilon(1) = 1$

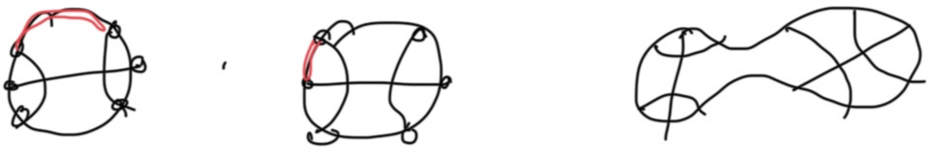
(2) $\delta(1) = 1 \otimes 1$

(3) $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$

(4) $\delta(ab) = \delta(a)\delta(b)$

Bialgebra of chord diagrams

product of chord diagrams

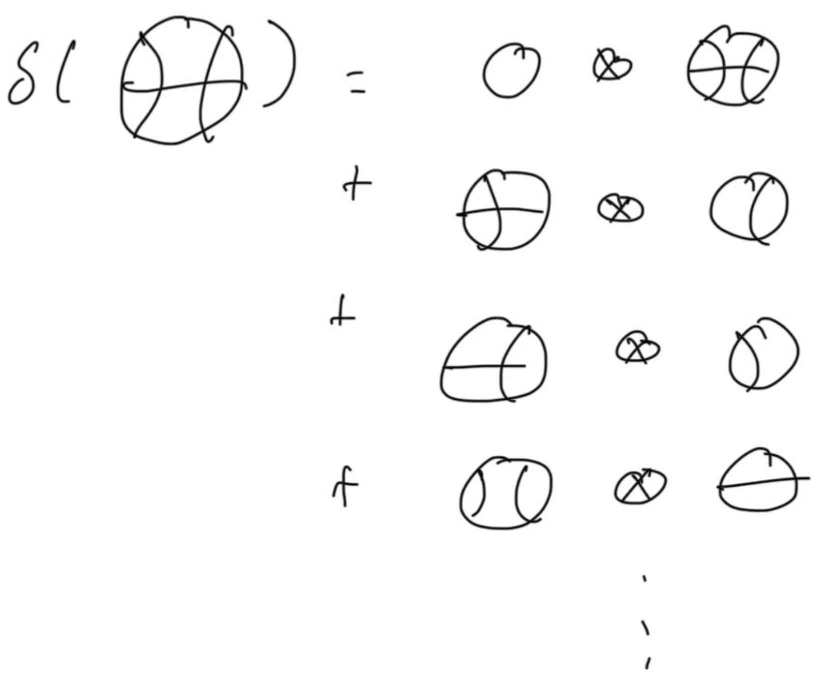


$$\mu: A^{fr}_m \otimes A^{fr}_n \longrightarrow A^{fr}_{m+n}$$

The product is well defined mod 4T

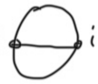
Coproduct

$$\delta: A^{fr}_n \longrightarrow \bigoplus_{k+l=n} A^{fr}_k \otimes A^{fr}_l$$



is well defined mod 4T

Lie Algebra weight systems

$\varphi_g($  $)$

$= \sum_{i=1}^{m} e_i e_i^*$ where the values of i span the elements of the representation of a Lie group; e_i is basis of g

e_i^* is dual basis of g

Why Lie groups? $\left.\begin{matrix}\text{STU}\\ \text{IHX}\end{matrix}\right\}$ relation $\rightsquigarrow$ Jacobi

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

$Sl(2)$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$[H,E] = 2E$

$[H,F] = -2F$ and $H^* = \frac{1}{2}H \quad E^* = F \quad F^* = E$

$[E,F] = H$

Casimir element: $C = \frac{1}{2}HH + EF + FE$

$\varphi_{sl_2}($  $) = \sum_{ij} e_i e_j e_i^* e_j^*$

$= (c-2)c$

End:

more Lie algeb weight syst.

Next time: Kontsevich