Quantum ~~invariants~~ talk 3

Ribbon Hopf algebras, Quantum sl_2 $(U_q(sl_2))$ and associated link invariants.

quasi-triangular Hopf algebra $\left\{ \begin{array}{l} \{\text{Hopf algebra } \cdot, \Delta, \varepsilon, i, S\} \\ + \\ \{\text{a universal } R\text{-matrix } R \in A \otimes A\} \end{array} \right.$

+

$\left\{ \begin{array}{l} v \in A \text{ which is a square root} \\ \text{of } S(u)u \text{ where} \\ u = \sum S(\beta_i)\alpha_i \\ (R = \sum \alpha_i \otimes \beta_i). \end{array} \right\}$

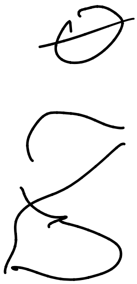
$\to$ ribbon Hopf algebra.

$\Rightarrow$ link invariants.

$U_q(sl_2) \to$ Kauffman bracket, Jones and Alexander polynomial.

Two ways of getting tangle invariants:

1. "TQFTs" $Z: \mathrm{Tang}^{or,fr} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$



$\emptyset$

$Z(\cdot) = V / \mathbb{C}$

link is a map $\emptyset \longrightarrow \emptyset$

$\rightsquigarrow \mathbb{C} \longrightarrow \mathbb{C}$

2. State sum invariants: you label your tangle by a bunch of things, and then you sum over all possible configurations.

§ Def. A Hopf algebra A over $\mathbb{C}$ is an algebra with unit 1 and maps

1. $\Delta : A \longrightarrow A \otimes A$ "comultiplication"

which is

1. coassociative
2. $\mathbb{C}$-algebra homomorphism

| counital with counit $\varepsilon : A \to \mathbb{C}$

$$\Delta(ab) = \Delta(a)\Delta(b)$$

$$(x \otimes y) \cdot (a \otimes b) = xa \otimes yb$$

2. Antipode map $S : A \to A$.

antihomomorphism: $S(ab) = S(b)S(a)$

It satisfies a compatibility condition:

if $x \in A$, $\Delta(x) = \sum x_i \otimes y_i$ then

$$\sum S(x_i) y_i = \sum x_i S(y_i)$$
$$= \varepsilon(x) \cdot 1.$$

Example: G finite group, $\mathbb{C}G$ is a Hopf algebra

comultiplication $\Delta(g) = g \otimes g$

$$S(g) = g^{-1} \qquad \varepsilon(g) = 1.$$

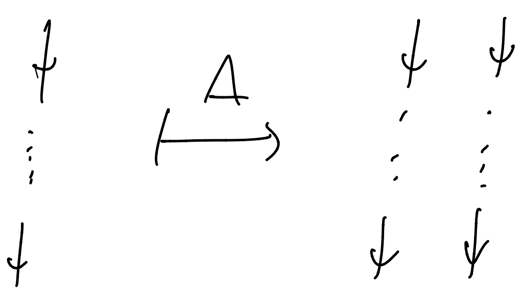
Remark

if V, W are G-representations, they are $\mathbb{C}G$-module. $V \otimes W$ is $\mathbb{C}G \otimes \mathbb{C}G$-module

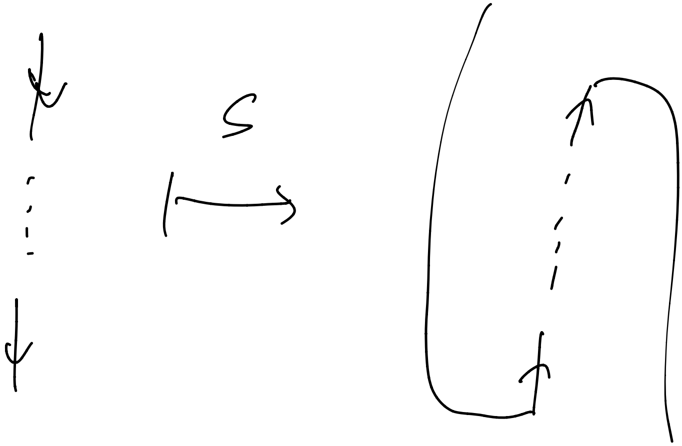
Via the comultiplication $\mathbb{C}G \to \mathbb{C}G \otimes \mathbb{C}G$, it becomes a $\mathbb{C}G$-module.

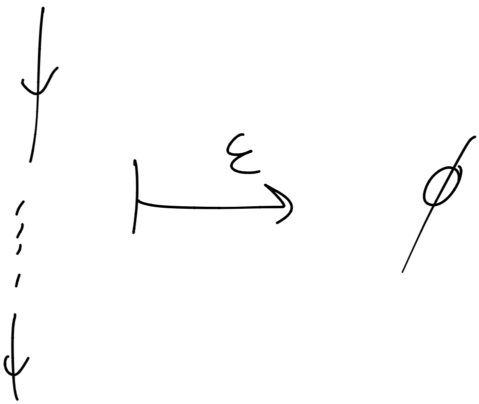
$$g.(v \otimes w) = gv \otimes gw = \Delta(g)\cdot(v \otimes w).$$

Remark: Tangles have some kind of Hopf algebra structure.

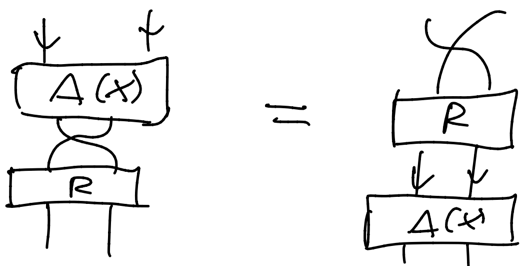


double a strand.



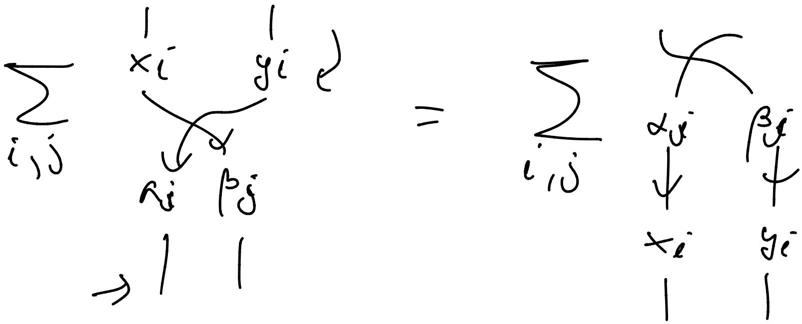


Definition. A quasitriangular Hopf algebra is a Hopf algebra A equipped with an element $R \in A \otimes A$ $\left(R = \sum_i \alpha_i \otimes \beta_i\right)$ satisfying a bunch of relations:

1. 

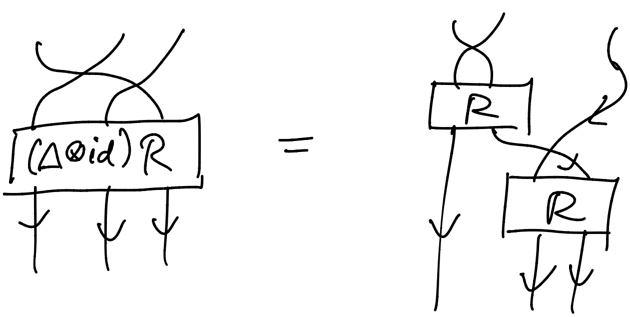
$\forall x \in A$

this means: $\Delta(x) = \sum x_i \otimes y_i$



$$\sum_{i,j} y_i\alpha_j \otimes x_i\beta_j = \sum_{i,j} \alpha_j x_i \otimes \beta_j y_i .$$

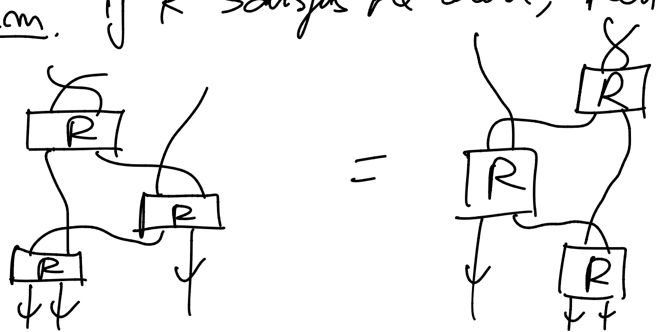
2:



$$\left(\sum \alpha_i\otimes\beta_i\otimes 1\right)\cdot\left(\sum 1\otimes\alpha_i\otimes\beta_i\right)$$

$$(\Delta\otimes id)\,R = (\Delta\otimes id)\left(\sum \alpha_i\otimes\beta_i\right)$$
$$= \underline{\sum \Delta(\alpha_i)\otimes\beta_i}$$

<u>Thm.</u> if R satisfies the above, then



Yang-Baxter equation in $A\otimes A\otimes A$!

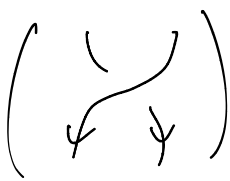
Remark: if V is a representation of A, then $V \otimes V$ is an $A \otimes A$-module.

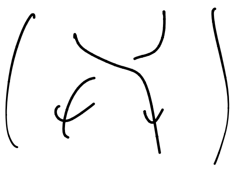
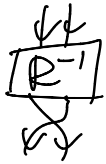
But $R \in A \otimes A$ so it acts on $V \otimes V$

$\Rightarrow$ it induces a linear map $R : V \otimes V \to V \otimes V$.

Because R satisfies Yang-Baxter in $A \otimes A \otimes A$, $R : V \otimes V \to V \otimes V$ also satisfies the Y-B for vector spaces.

Can try to build an invariant $Q^{A,*}$ of tangles:

$Q^{A;*}$  $=$  ;

$Q^{A;*}$  $=$ 

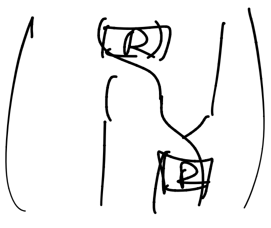
$Q(\cap) = \cap$

$Q(\cap) = \cap$

$Q(\cup) = \cup$

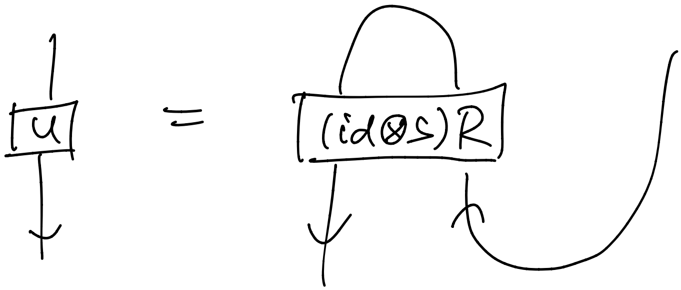
$Q(\cup) = \cup$.

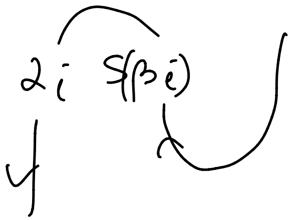
$Q(\downarrow) = \downarrow$; $Q(\uparrow) = \uparrow$;

Q  $\rightsquigarrow$ 

Def $u = \sum S(\beta_i)\,\alpha_i \in A.$

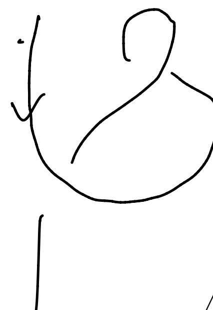
$(R = \sum \alpha_i \otimes \beta_i)$



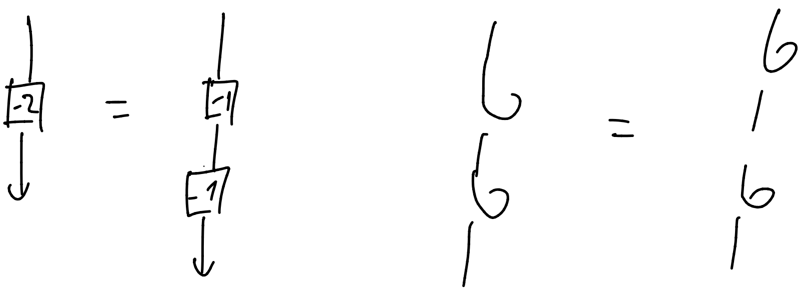
$=\sum$ 

$= u.$

$Q^{A;*}\Big($ 

$\Big) = Q^{A;*}\Big($  $\Big)$

$= \underline{S(u)\,u}$



So if $Q^{A,v}\left(\begin{smallmatrix} 6 \\ | \end{smallmatrix} \right) = v$, you would want

$$v^2 \stackrel{?}{=} S(u)\cdot u.$$

Def. A ribbon Hopf algebra is a quasitriangular Hopf algebra with an element $v \in A$ st:

1. v central in A
2. $v^2 = S(u)u$

$u = \sum S(\beta_i)\alpha_i$

where $R = \sum \alpha_i \otimes \beta_i$

..... + other conditions.

$\Rightarrow$ can define the full invariant.

This invariant $Q^{A,*}$, on a link diagram with l components, gives an element in

$$(A/I)^{\otimes l}.$$

If V is a representation of A,

then an element $Q^{A,*}(L) \in (A/I)^{\otimes \ell}$

defines a rep $V^{\otimes \ell} \to V^{\otimes \ell}$.

To get a number, can take the trace.

§ Quantum groups

Def. $sl_2 = \{ M \in Mat_{2\times 2}(\mathbb{C}) \mid \operatorname{tr} M = 0 \}$.

Lie algebra with $[X,Y] = XY - YX$.

This is generated by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$[H,E] = 2E, \quad [H,F] = -2F, \quad [E,F] = H.$

Associated to this the is $U(sl_2)$

this is an algebra over $\mathbb{C}$ with unit 1 and

generators E, F, H satisfying.

$\underline{HE - EH = 2E}$, $HF - FH = -2F$, $EF - FE = H$.

Def. Let $q \in \mathbb{C} \setminus \{0\}$, not a root of unity.

The quantum group $U_q(sl_2)$ is the

$\mathbb{C}$-algebra generated by symbols K, K^{-1}, E, F

st.

1. $K \cdot K^{-1} = 1$, $K^{-1} \cdot K = 1$.

$\boxed{KE = qEK,}$ $KF = q^{-1} FK$

$$EF - FE = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}}$$

Think: $K = q^{H/2}$ $q = e^{t}$.

Can rey $U_q(sl_2) \to U(sl_2)[[\hbar]]$

by sending $q \mapsto e^{\hbar}$.

Can do some calculation:

$$KE = e^{\hbar H/2} E = \sum_{i=0}^{\infty} \frac{\hbar^n}{2^n n!} \cdot H^n E.$$

$$= H \cdot e^{\hbar\left(\frac{E+2}{2}\right)}$$

$HE - EH = 2E$;

$$= H \cdot e^{\hbar E/2} \cdot e^{\hbar}$$

$$= qEK.$$

$$H^n E = E(H+2)^n.$$

$$EF - FE = \frac{q^{H/2} - q^{-H/2}}{q^{1/2} - q^{1/2}} = [H].$$

$$\lim_{q \to 1} \frac{q^{H/2} - q^{-H/2}}{q^{1/2} - q^{1/2}} = H.$$

$U_q(sl_2)$ is a ribbon Hopf algebra.

Hopf algebra structure:

$U(sl_2)$ is a Hopf algebra, $\Delta(x) = 1\otimes x + x \otimes 1$, $S(x) = -x$, $\varepsilon(x) = 0$

$$\Delta(k^{\pm 1}) = k^{\pm 1} \otimes k^{\pm 1} \qquad S(k^{\pm 1}) = k^{\mp 1}$$

$$\Delta(E) = E \otimes K + 1 \otimes E \qquad S(E) = -E k^{-1}$$

$$\Delta(F) = F \otimes 1 + K^{-1} \otimes F \qquad S(F) = -kF$$

$$\varepsilon(k^{\pm 1}) = 1; \quad \varepsilon(E) = \varepsilon(F) = 0.$$

Quasitriangular, via the R-matrix:

$$R = q^{H \otimes H/4} \exp_q\left((q^{1/2} - q^{-1/2}) E \otimes F \right).$$

$$e^{t H \otimes H/4} \in U(g) \otimes U(g) \, [[t]]$$

can think of it as living in $U_q(sl_2) \otimes U_q(sl_2)$.

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/4}}{[n]!} x^n$$

$[n]$ is the "quantum integer"

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$$

and $[n]! = [1]\cdot[2]\cdot \ldots \cdot [n]$.

$u = \sum S(\beta_i)\alpha_i$ where $R = \sum \alpha_i \otimes \beta_i$

wanted a square root of $S(u)u$.

icular:

$$u = q^{-H^2/4} \sum_{n=0}^{\infty} q^{3n(n-1)/4} \frac{(q^{-1/2} - q^{1/2})}{[n]!} F^n K^{-n} E^n$$

and $v = K^{-1} u$.

Theorem $(U_q(sl_2), R, v)$ is a Ribbon Hopf algebra.

A remark on my theory

The irreps of sl_2 $(U(sl_2))$ are given by V_n, $n \geq 1$, $V_n \doteq \mathbb{C}^n$ and

$$\rho_{V_n}(E) = \begin{pmatrix} 0 & [n-1] & & & \\ & 0 & [n-2] & & \\ & & \ddots & \ddots & \\ & & & 0 & [1] \\ & & & & 0 \end{pmatrix}$$

$$\rho_{V_n}(F) = \begin{pmatrix} 0 & & & & \\ [1] & 0 & & & \\ & [2] & \ddots & & \\ & & \ddots & \ddots & \\ & & & [n-1] & 0 \end{pmatrix}$$

$$\rho_{V_n}(\cancel{H}\, K) = \begin{pmatrix} q^{\frac{n-1}{2}} & & & & \\ & q^{\frac{n-3}{2}} & & & \\ & & q^{\frac{n-5}{2}} & & \\ & & & \ddots & \\ & & & & q^{\frac{-(n-1)}{2}} \end{pmatrix}$$

$K = q^{H/2}$

$\Rightarrow$ get link invariants "quantum (sl_2, V_n)"

$$\in \mathbb{Z}[q^{\pm 1/2}]$$

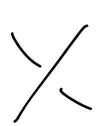
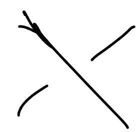
$V_n^\vee \cong V_n$ as reps, this does not depend on orientation.

Ex $\underline{n=2}$.

$$\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\rho(K) = \begin{pmatrix} q^{1/2} & \\ & q^{-1/2} \end{pmatrix}.$$

$$\rho(E^2) = 0, \quad \rho(F^2) = 0.$$

$q^{1/4}\, Q\Big($  $\Big) - q^{-1/4}\, Q\Big($  $\Big)$

$$\rightsquigarrow \quad = (q^{1/2} - q^{-1/2})\, Q\big(\,)\,(\,\big).$$

$$q^{1/4} R - q^{-1/4} R^{-1} = \left(q^{1/2} - q^{-1/2}\right) \mathrm{id}_{V \otimes V}$$

$$Q(L) = (-1)^{\#L + f(L)} \langle L \rangle \Big|_{A = q^{1/4}}$$

# of components

sum of framings of components.

Alexander polynomial

To get it, you need $U_q(sl_2)$ at a root of unity.

Def. Let $\xi = e^{2\pi i/r}$ be a root of unity.

We define $U_\xi(sl_2)$ to be:

- generated by K, K^{-1}, F, E
- same relations as $U_q(sl_2)$

$+ \; E^r = F^r = 0.$

Before:

$$R = q^{H\otimes H/4} \exp_q\left(\left(q^{1/2} - q^{-1/2}\right) E\otimes F\right)$$

$$\overset{(<r)}{=} \sum_{i=0}^{r-1} (\text{same thing as before}).$$

$\rightsquigarrow$ quasi-triangulated H. A.

$$u = \sum S(\beta_i)\alpha_i \quad \ldots$$

$$v = K^{n-1} u \qquad (v^2 = S(u)u).$$

Reg Rep is different: it has a bunch of irreps on $\mathbb{C}^n$, parametrised by $\lambda \in \mathbb{C}$.

$$\rho_\lambda(E) = \begin{pmatrix} 0 & [n-1+\lambda] & & & \\ & 0 & [n-2+\lambda] & & \\ & & \ddots & \ddots & \\ & & & & [1+\lambda] \\ & & & & 0 \end{pmatrix}$$

$$\rho_\lambda(F) = \begin{pmatrix} 0 & & & & \\ [1] & 0 & & & \\ & [2] & \ddots & & \\ & & \ddots & & \\ & & & [n-1] & 0 \end{pmatrix}$$

$$\rho_\lambda(K) = \begin{pmatrix} q^{(n-1+\lambda)/2} & & & \\ & q^{(n-3+\lambda)/2} & & \\ & & \ddots & \\ & & & q^{(-n+1+\lambda)/2} \end{pmatrix}$$

$r=2$: we get

$$\rho_t(E) = \begin{pmatrix} 0 & [1+t] \\ 0 & 0 \end{pmatrix}$$

$$\rho_t(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\rho_t(K) = \begin{pmatrix} \xi^{(1+t)/2} & \\ & \xi^{(-1+t)/2} \end{pmatrix}.$$

For this particular representation you get some polynomial in $\mathbb{Z}[t]$

$\rightsquigarrow$ Alexander polynomial.