Notes for Quantum Invariants

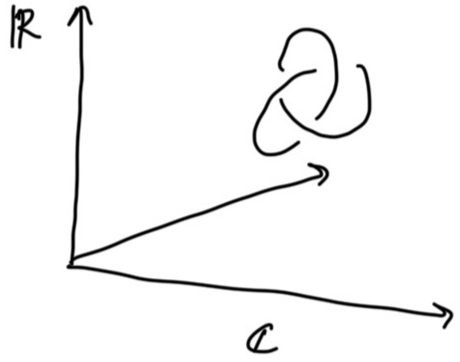
20 March

Kontsevich Integral

Goal: Introduce the Kontsevich integral.

:

1.) Knots are going to live in $\mathbb{C} \times \mathbb{R}$



Historical Rmks:

* The Kontsevich integral appeared as a tool to prove the Fundamental theorem of Vassiliev invariants.

Recall:

Fundamental Theorem:

$$\mathcal{W} \cong \bigoplus_{n=0}^{\infty} \mathcal{V}_n / \mathcal{V}_{n-1}$$

what does this say?

↳ There is a map ($\bar{\alpha}_n$) that identifies any vassiliev invariant to a weight system.

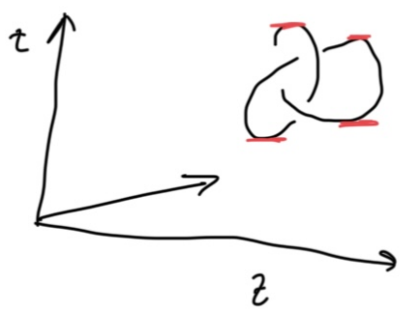
→ <u>Example:</u>

to any 2^{nd} order vassiliev inv. there will be some linear combination of weight systems. (order 2) 

→ <u>Motivating Example</u> (to construct the integral).

Morse knots, links, tangles.

↳

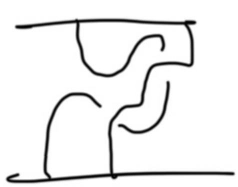


$t \in \mathbb{R}$

$z \in \mathbb{C}$

→ critical points non-degenerate in t.

Link



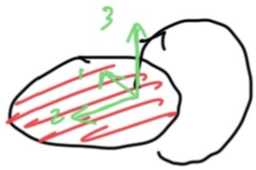
Tangle

(Self) Linking Number

$lk(A,B) = \#$ times A loops round B
$= 1$

Defn:

$lk(A,B)$: Choose an oriented disc D_A s.t A is the boundary



$\rightsquigarrow$ $lk(A,B)$ = intersection no. between D_A and B.

$\hookrightarrow$ Assign ± 1 depending on orientation

: if (e_1, e_2, e_3) defines positive orientation on $\mathbb{R}^3$

Integral formula:
Let $\alpha, \beta : S^1 \to \mathbb{R}^3$

$$lk(A,B) = \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{(\beta(v) - \alpha(u)) \cdot (du \times dv)}{|\beta(v) - \alpha(u)|^3}$$

Geometrically: $\#$ Normalized vector connecting a point on A & B goes around the sphere $S^1 \times S^1$.

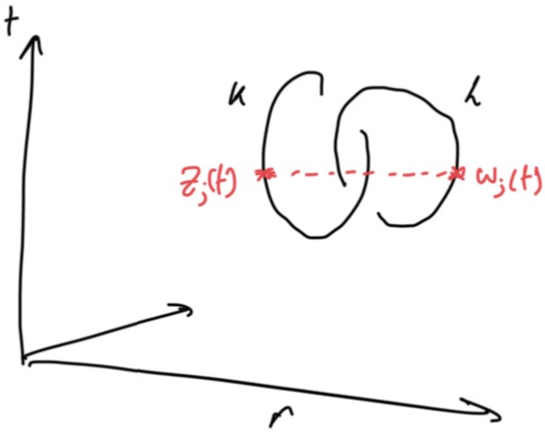
linking number is a vassiliev invariant. [EXAMPLE]

$\vec{v}$ () = v () − v ()

$\therefore$ ↳ lk () = $\underline{lk}$ ()

linking number is (at least) a vassiliev invariant of type I.

Morse link

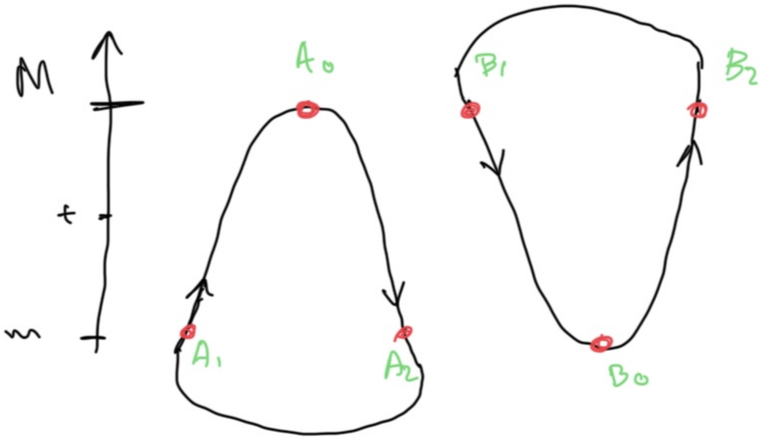


$$lk(k, L) = \frac{1}{2\pi i} \int \underbrace{\sum_j (-1)^j \frac{d(z_j(t) - w_j(t))}{z_j(t) - w_j(t)}}$$

already kinda resembles Gaussian linking int.

$t \in \mathbb{R}$, $r \in \mathbb{C}$

1.) $\displaystyle \sum_j (-1)^j \frac{d(z_j(t) - w_j(t))}{z_j(t) - w_j(t)} \longrightarrow$ <u>integer.</u>



The config space of all horizontal coords joining K, L is a 1D manifold

→ Disjoint union of $\bigcirc$

$$A_1B_0 \xrightarrow{+1} A_0B_1 \xrightarrow{-1} A_2B_0 \xrightarrow{+1} A_0B_2 \xrightarrow{-1} A_1B_0 \qquad \bigcirc = 0$$

note: $(-1)^{\downarrow j}$ →

Integral formula counts no. of complete turns of horizontal chord.

NOTICE:

1.) The value in the integrand is unchanged under horizontal deformation

2.) Reduction to combinatorial formula.

* The Kontsevich integral keeps track of how finite sets of horizontal chords rotate when moved upward.

Naive approach :⇒ KI is the monodromy of the KZ connection in the complement of the union of diagonals in $\mathbb{C}$

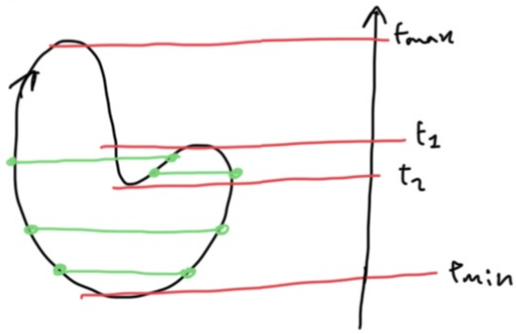
Def:

the Kontsevich Integral

$$Z(K) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{t_{min} < t_m < \cdots t_1 < t_{max}} \sum_{P=\{(z_j, z_j')\}} (-1)^{\downarrow P} D_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'}$$

unpack

Ex. $m = 2$



Picking $(z_j, t_j), (z_j', t_j)$

if $z_j(t_j)$ and $z_j'(t_j)$ are continuous.

$P = \{(z_j, z_j')\}$

$t_2 < t < t_1$

36  , 

A pairing is a way to inscribe chord diagrams at different simplexes.

$t_{min} < t < t_2$

1 

$\downarrow P$: no. points oriented downward in pairing.

Basic properties

- $Z(K)$ converges for Morse knots
- Invariant under deformations in the class of Morse knots.

→ what happens when we add new critical points?

$Z($  $) = Z(H) \cdot Z($  $)$

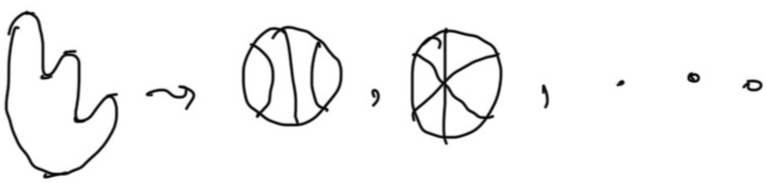
$H :=$  Define a 'genuine' knot invariant

$$I(K) = \frac{Z(K)}{Z(H)^{c/2}}$$

c = number of critical points in K

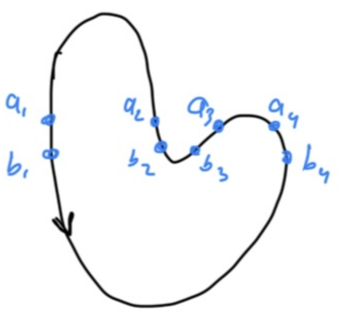
Division in the algebra $\mathcal{A}$ of chord diagrams.

4 

6 

Example calc.

Calculate coefficient of  in $Z(H)$



$\rightsquigarrow$ 16 pairs.

The 16 pairs are the differential forms:

$$(-1)^{j+k+l+m} \, d\ln a_{jk} \wedge d\ln b_{lm}$$

where $a_{jk} = a_k - a_j$

$(jk) \in \{(13), \dots\} = A$

$(lm) \in \{(12), \dots\} = B$

$$\text{coeff} = \frac{1}{(2\pi i)^2} \int_{\Delta} \sum_{(jk)\in A} \sum_{(lm)\in B} (-1)^{j+k+l+m} \, d\ln a_{jk} \wedge d\ln b_{lm}$$

$$= -\frac{1}{4\pi^2} \int d\ln \frac{a_{14}\, a_{23}}{a_{13}\, a_{24}} \wedge d\ln \frac{b_{12}\, b_{34}}{b_{13}\, b_{24}}$$

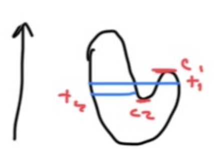
$\rightarrow$ $u = \dfrac{a_{14}\, a_{23}}{a_{13}\, a_{24}}$ $\quad v = \dfrac{b_{12}\, b_{34}}{b_{13}\, b_{24}}$

$$\frac{1}{4\pi^2} \int_{\Delta'} d\ln u \wedge d\ln v$$

$\rightsquigarrow$ domain: $\rightarrow$

$c_2 < t_1 < c_1$

$c_2 < t_2 < t_1$



* Need to understand the change of variables.

$$\frac{1}{4\pi^2} \int_{\Delta'} d\ln u \wedge d\ln v$$

$$= \frac{1}{4\pi^2} \int_0^1 \left(\int_{1-u}^1 d\ln v \right) \frac{du}{u}$$

$$= -\frac{1}{4\pi^2} \int_0^1 \ln(1-u) \frac{du}{u} \longrightarrow \underline{\frac{1}{24}}$$

$\underline{\frac{1}{24}}$ $\otimes$

Invariance

Deformations of Morse knots; sequence of deformations of 3 types.

* orientation-preserving reparam
* horizontal deformations
* movements of critical points

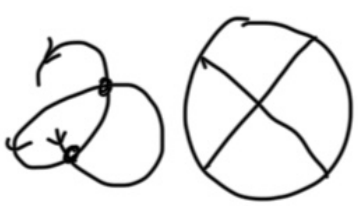
Kontsevich integrals for tangles

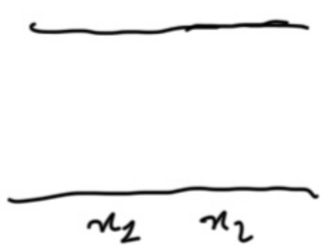
$Z($  $)$

⇝ ~~completed~~ instead of algebra $\hat{A}$ of chord diagrams

we have: graded completion of vect space of tangle chord diagrams

→ Tangle chord diagram

 ,



The universal Vassiliev invariant

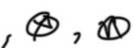
some chord diagram satisfying 4T 1T

Theorem:

Let w be an unframed weight system of order n. Then there exists a Vassiliev invariant of order n whose symbol is w.

Proof:

Recall:

$$I(K) = \frac{Z(K)}{Z(H)^{c/2}} \rightsquigarrow \bigcirc, $$  $\dots$

$K \mapsto w(I(K)) \leftarrow$ this is the knot inv. we are talking about.

Let D chord diagram order n.

K_D singular knot w/ D

$$I(K_D) = D + \dots$$

$Z(H)^{c/2}$ starts with the unit of $\mathcal{A}$.

[unit : $\bigcirc$ $\rightsquigarrow$ chord diagrams admit a Hopf algebra]

$$Z(K_D) = D + \dots$$

If $n=0$

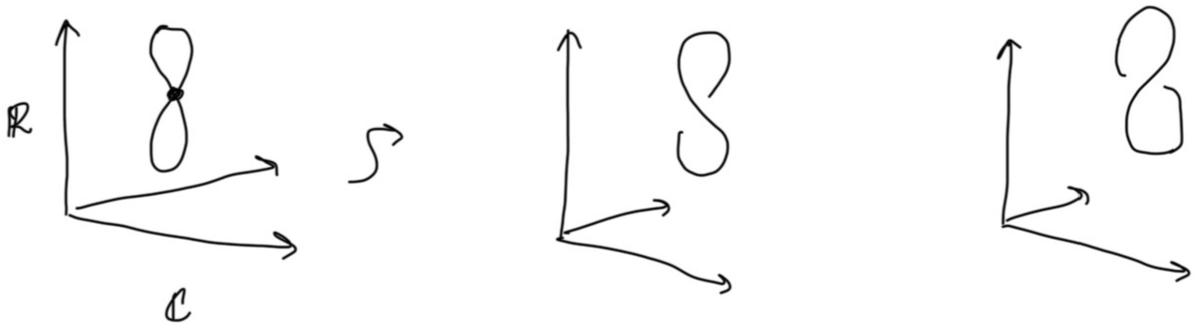
$\hookrightarrow$ $D = \bigcirc$

$$Z(K_D) = \bigcirc \quad \nu$$

If $n=1$

↪ $D =$ 

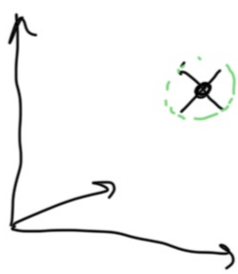
$Z(K_D) \rightsquigarrow K_D$ = knot w/ 1 double point.



⇝ there is only one pair

$P(\{z_j, z_j'\})$ 

for arbitrary knots with singular point



consider neighbourhood of singularity (ε)

→ $Z(K_\pm^\varepsilon) = Z_\pm^1$ (all chords in neighbourhood)

$Z_\pm^n$ (all chords the rest)

⇝ $Z_\pm^1 =$ 

for knots of $1 > n$.

→ Deform knots s.t. we look at product of n tangles w/ one double point?