The coloured Jones polynomial and the volume conjecture

References : "An introduction to the volume conjecture" H. Murakami

"Quantum invariant of Knots and 3-manifolds" V. G. Turaev

YANG - BAXTER OPERATOR :

Consider :
- V a N-dim. $\mathbb{C}$-vec. space
- $R \in \operatorname{End}(V \otimes V)$
- $\mu \in \operatorname{End}(V)$
- $a, b \in \mathbb{C} \setminus \{0\}$

DEF: a quadruple (R, μ, a, b) is an enhanced Yang-Baxter operator if it satisfies the following:

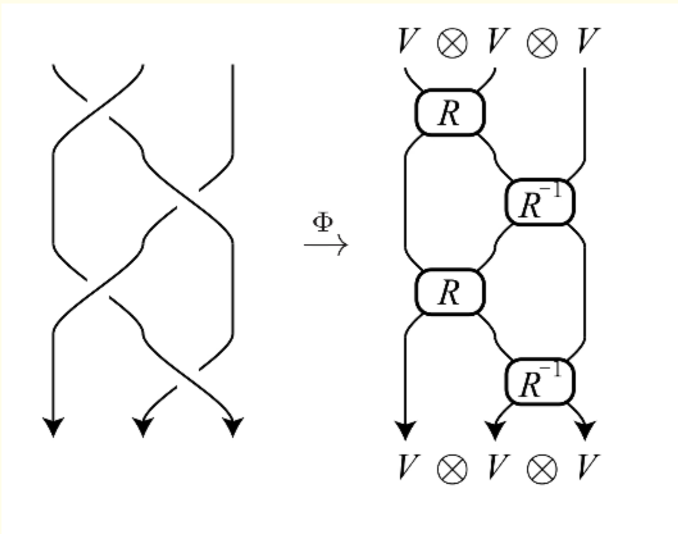
(i) YBE : $(R \otimes id_V)(id_V \otimes R)(R \otimes id_V) = (id_V \otimes R)(R \otimes id_V)(id_V \otimes R)$

(ii) $R(\mu \otimes \mu) = (\mu \otimes \mu) R$

(iii) $Tr_2(R^{\pm}(id_V \otimes \mu)) = a^{\pm 1} b \, id_V$

Recall that $Tr_K : \operatorname{End}(V^{\otimes K}) \longrightarrow \operatorname{End}(V^{\otimes (K-1)})$.

Given a braid $\beta \in B_m$, we associate to β a homomorphism $\phi(\beta) \in \operatorname{End}(V^{\otimes m})$:



DEF: for $\beta \in B_m$ we define $T_{(R,\mu,a,b)}(\beta) \in \mathbb{C}$ as :

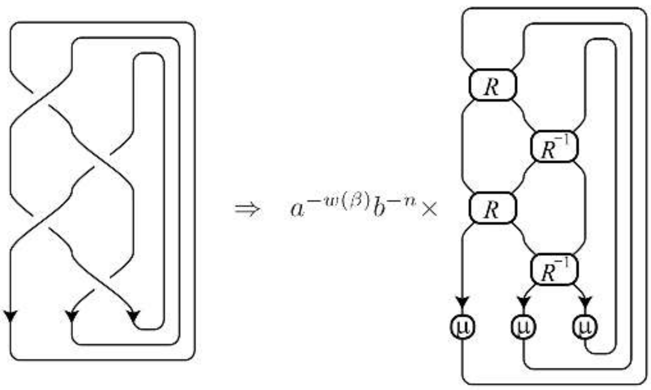
$$T_{(R,\mu,a,b)}(\beta) = a^{-w(\beta)} b^{-m} \, Tr_1(Tr_2(\dots(Tr_m(\phi(\beta)\mu^{\otimes m}))\dots))$$

with $w(\beta)$ = sum of exponents of β.

ex:

$$T_{(R,\mu,a,b)}(\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1})$$
$$= b^{-3}\,\mathrm{Tr}_1(\mathrm{Tr}_2(\mathrm{Tr}_3((R\otimes \mathrm{Id}_V)(\mathrm{Id}_V\otimes R^{-1})(R\otimes \mathrm{Id}_V)(\mathrm{Id}_V\otimes R^{-1})(\mu\otimes\mu\otimes\mu))))$$

since $w(\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}) = +1-1+1-1=0$ (Figure 9).



RMK: μ plays the role of uv^{-1} in the $Q^{A;*}$ construction

THM. (Turaev): if β and β' have isotopic closures, then $T_{(R,\mu,a,b)}(\beta) = T_{(R,\mu,a,b)}(\beta')$

The colored Jones polynomial

Consider the Lie algebra $sl_2(\mathbb{C})$ and its N-dimensional irreducible representation V_N. The quantum $(sl_2(\mathbb{C}), V_N)$ invariant is the colored Jones polynomial $J_N(L;q)$.

Explicit construction:

Consider $V := \mathbb{C}^N$ and $R: V\otimes V \longrightarrow V\otimes V$ the R-matrix defined by

$$R(e_k\otimes e_l) = \sum_{i,j=0}^{N-1} R^{ij}_{kl}(e_i\otimes e_j)$$

$$R^{ij}_{kl} := \sum_{m=0}^{\min(N-1-i,\,j)} \delta_{l,\,i+m}\,\delta_{k,\,j-m}\,\frac{\{l\}!\,\{N-1-k\}!}{\{i\}!\,\{m\}!\,\{N-1-j\}!} \times q^{(i-(N-1)/2)(j-(N-1)/2)-m(i-j)/2-m(m+1)/4}$$

where $\{m\} := q^{m/2} - q^{-m/2}$ and $\{m\}! := \{1\}\ldots\{m\}$.

Let $\mu: V\longrightarrow V$ be the homomorphism given by

$$\mu(e_j) = \sum_{i=0}^{N-1} \mu^i_j\, e_i \qquad \text{with} \qquad \mu^i_j := \delta_{i,j}\, q^{(2i-N+1)/2}$$

DEF: the quadruple $(R, \mu, q^{(N^2-1)/4}, 1)$ is an enhanced YB operator. The N-dimensional colored Jones polynomial $J_N(L;q)$, for a link L, is defined as:

$$J_N(L;q) := T_{(R,\mu,q^{(N^2-1)/4},1)}(\beta)\times\frac{\{1\}}{\{N\}},$$

with β braid s.t. $\hat{\beta}\cong L$.

LEMMA: we have the following skein relation:

$$q\, J_2(L_+;q) - q^{-1} J_2(L_-;q) = (q^{1/2} - q^{-1/2})\, J_2(L_0;q)$$

Consequence: $J_2(L;q) = (-1)^{\#L-1} V_L(q^{-1})$, with $\#L$ = number of components of L.

Proof: we have

$$R = \begin{pmatrix} q^{1/4} & 0 & 0 & 0 \\ 0 & q^{1/4-3/4} & q^{-1/4} & 0 \\ 0 & q^{-1/4} & 0 & 0 \\ 0 & 0 & 0 & q^{1/4} \end{pmatrix} \qquad \mu = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}$$

Note that L_+, L_-, L_0 can be represented by $\beta\sigma_i\beta'$, $\beta\sigma_i^{-1}\beta'$ and $\beta\beta'$, so

$$\begin{aligned}
&\left(qJ_2(L_+;q) - q^{-1}J_2(L_-;q)\right) \times \frac{\{2\}}{\{1\}} \\
=&q \times q^{-3(w(\beta\beta')+1)/4}\,\mathrm{Tr}_1(\mathrm{Tr}_2(\cdots(\mathrm{Tr}_n(\Phi(\beta\sigma_i\beta')\mu^{\otimes n})))) \\
&\quad - q^{-1} \times q^{-3(w(\beta\beta')-1)/4}\,\mathrm{Tr}_1(\mathrm{Tr}_2(\cdots(\mathrm{Tr}_n(\Phi(\beta\sigma_i^{-1}\beta')\mu^{\otimes n})))) \\
=&q^{-3w(\beta\beta')/4} \\
&\times\left(q^{1/4}\,\mathrm{Tr}_1(\mathrm{Tr}_2(\cdots(\mathrm{Tr}_n(\Phi(\beta\sigma_i\beta')\mu^{\otimes n})))) - q^{-1/4}\,\mathrm{Tr}_1(\mathrm{Tr}_2(\cdots(\mathrm{Tr}_n(\Phi(\beta\sigma_i^{-1}\beta')\mu^{\otimes n}))))\right) \\
=&q^{-3w(\beta\beta')/4} \\
&\times\Big\{\mathrm{Tr}_1(\mathrm{Tr}_2(\cdots(\mathrm{Tr}_n(\Phi(\beta)(\mathrm{Id}_V^{\otimes(i-1)} \otimes q^{1/4}R \otimes \mathrm{Id}_V^{\otimes(n-i-1)})\Phi(\beta')\mu^{\otimes n})))) \\
&\quad - \mathrm{Tr}_1(\mathrm{Tr}_2(\cdots(\mathrm{Tr}_n(\Phi(\beta)(\mathrm{Id}_V^{\otimes(i-1)} \otimes q^{-1/4}R^{-1} \otimes \mathrm{Id}_V^{\otimes(n-i-1)})\Phi(\beta')\mu^{\otimes n}))))\Big\}
\end{aligned}$$

(from (2.3))

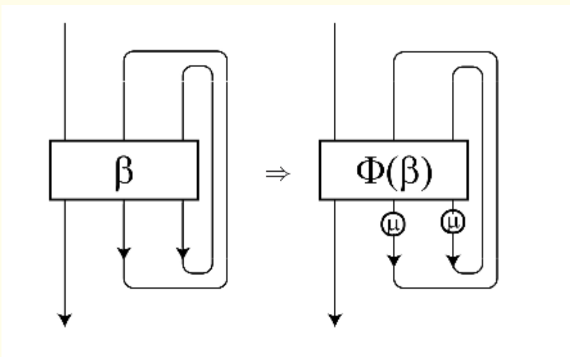
$$\begin{aligned}
=&q^{-3w(\beta\beta')/4}(q^{1/2} - q^{-1/2})\,\mathrm{Tr}_1(\mathrm{Tr}_2(\cdots(\mathrm{Tr}_n(\Phi(\beta\beta')\mu^{\otimes n})))) \\
=&(q^{1/2} - q^{-1/2})J_2(L_0;q) \times \frac{\{2\}}{\{1\}},
\end{aligned}$$

completing the proof. □

Example: the figure-eight knot

Consider $\beta = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$. Observe that, by Schur's lemma, $\mathrm{Tr}_2(\mathrm{Tr}_3(\Phi(\beta)(\mathrm{id}\otimes\mu\otimes\mu)))$ is a scalar multiple of the identity, $S\cdot \mathrm{id}_V \in \mathrm{End}(V)$.

Therefore $\mathrm{Tr}_1(\mathrm{Tr}_2(T_3(\Phi(\beta)\,\mu^{\otimes 3}))) = \mathrm{trace}(\mu)\cdot S$



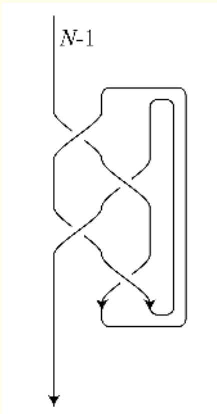
We have :

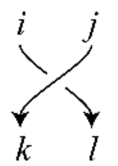
$$\begin{aligned}
T_{(R,\mu,q^{(N^2-1)/4},1)}(\beta) &= q^{-w(\beta)(N^2-1)/4}\,\mathrm{Tr}_1(\mathrm{Tr}_2(\mathrm{Tr}_3(\Phi(\beta)\mu^{\otimes 3})))\\
&= q^{-w(\beta)(N^2-1)/4}\,\mathrm{Tr}_1(S\,\mathrm{Id}_V)\\
&= q^{-w(\beta)(N^2-1)/4}\sum_{i=0}^{N-1} S\,q^{(2i-N+1)/2}\\
&= q^{-w(\beta)(N^2-1)/4}\frac{\{N\}}{\{1\}}S,
\end{aligned}$$

So $J_N(K;q) = q^{-w(\beta)(N^2-1)/4}\,\frac{\{N\}}{\{1\}}\,S \cdot \frac{\{1\}}{\{N\}} = S$ as $w(\beta)=0$.

We have to compute the scalar S:

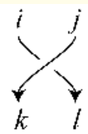
fix $\{e_0 \dots e_{N-1}\}$ basis of $\mathbb{C}^N$ $\Rightarrow$ label each arc with an e_i. Since we are doing $S\cdot \mathrm{id}$ we can start with any element. We take e_{N-1}



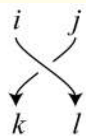
Recall that we have 

$\Rightarrow R^{ij}_{kl}$, $\Rightarrow (R^{-1})^{ij}_{kl}$

So we label the other arcs following two rules :

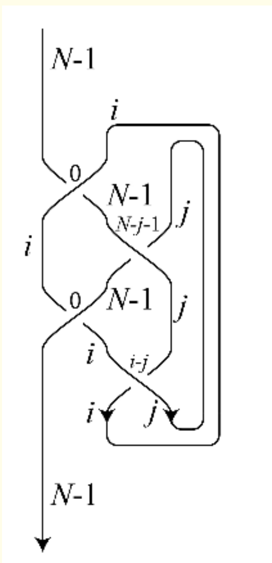
(i) 

$: i+j=k+l,\ l\geq i,\ k\leq j,$

(ii) 

$: i+j=k+l,\ l\leq i,\ k\geq j.$

We get the following :



We can now compute it :

$$J_N(E;q) = \sum_{i\geq j} R^{N-1,i}_{i,N-1}\,(R^{-1})^{N-1,j}_{N-1,j}\,R^{i,N-1}_{N-1,i}\,(R^{-1})^{i,j}_{i,j}\,\mu^j_j\,\mu^i_i$$
$$= \sum_{i\geq j}(-1)^{N-1+i}\frac{\{N-1\}!\{i\}!\{N-1-j\}!}{(\{j\}!)^2\{i-j\}!\{N-1-i\}!}$$
$$\times\, q^{(-i-i^2-2ij-2j^2+3N+6Ni+2Nj-3N^2)/4}.$$

This can be semplified into the following formula, due to Habiro and Lê :

$$L_N(K;q) = \frac{1}{\{N\}}\sum_{K=0}^{N-1}\frac{\{N+K\}!}{\{N-1-K\}!}$$

Volume conjecture

CONJECTURE : for any Knot K we have

simplicial volume

$$2\pi\lim_{N\to\infty}\frac{\log|J_N(K;\exp(2\pi\sqrt{-1}/N))|}{N} = \mathrm{Vol}(S^3\setminus K).$$

What is $\mathrm{Vol}(S^3\setminus K)$?

Consider a Knot K, then its complement can be decomposed using the JSJ decomposition as

$$S^3\setminus K = \Big(\bigsqcup_i H_i\Big)\sqcup\Big(\bigsqcup_j E_j\Big),$$

with H_i hyperbolic and E_j Seifert-fibered $\forall\, i,j$.

DEF : the SIMPLICIAL VOLUME of $S^3\setminus K$ is defined as

$$\mathrm{Vol}(S^3\setminus K) := \sum_{H_i}\text{hyp. vol.}(H_i)$$

Example : figure-eight Knot

Consider the formula for the colored Jones polynomial, we have

$$J_N(E;q) = \sum_{j=0}^{N-1}\prod_{k=1}^{j}\left(q^{(N-k)/2}-q^{-(N-k)/2}\right)\left(q^{(N+k)/2}-q^{-(N+k)/2}\right).$$

$q = \exp(2\pi\sqrt{-1}/N)$.

$$J_N\left(E;\exp(2\pi\sqrt{-1}/N)\right) = \sum_{j=0}^{N-1}\prod_{k=1}^{j} f(N;k)$$ with $f(N;K) = 4\sin^2(K\pi/N)$

Passing to log (+ extra steps) we get:

$$\lim_{N\to\infty} \frac{\log J_N\left(E; \exp(2\pi\sqrt{-1}/N)\right)}{N} = \lim_{N\to\infty} \frac{\log g(N; 5N/6)}{N}.$$

with $g(N;j) := \prod_{k=1}^{j} f(N;k)$

and then

$$\begin{aligned} \lim_{N\to\infty} \frac{\log g(N;5N/6)}{N} &= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log f(N;k) \\ &= 2 \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{5N/6} \log(2\sin(k\pi/N)) \\ &= \frac{2}{\pi} \int_0^{5\pi/6} \log(2\sin x)\, dx. \\ &= -\frac{2}{\pi} \Lambda(5\pi/6) \end{aligned}$$

Lobachevsky function

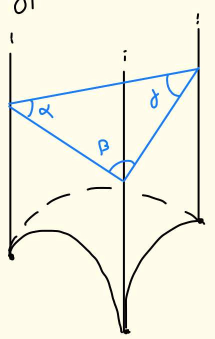
Using properties of Λ, we have:

$$2\pi \lim_{N\to\infty} \frac{\log J_N\left(E; \exp(2\pi\sqrt{-1}/N)\right)}{N} = 6\Lambda(\pi/3).$$

We now pass to hyperbolic geometry:

THM (Thurston): the complement of the figure-eight knot can be obtained by gluing two ideal hyperbolic regular tetrahedra.

FACT: the hyperbolic volume can be expressed using the function Λ



$$\mathrm{Vol}(\Delta(\alpha,\beta,\gamma)) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$$

Using the above, we have that for the figure-eight knot:

$$\mathrm{Vol}(S^3 \setminus K) = 2\,\mathrm{Vol}(\Delta(\pi/3, \pi/3, \pi/3)) = 6\Lambda(\pi/3)$$

And we conclude as:

$$2\pi \lim_{N\to\infty} \frac{\log J_N(K; \exp(2\pi\sqrt{-1}/N))}{N} = 6\Lambda(\pi/3) = \mathrm{Vol}(S^3 \setminus K)$$