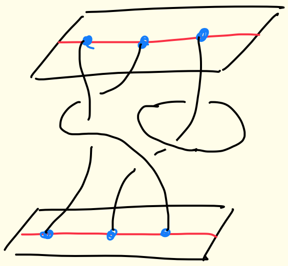
Operator invariants of tangles

30/01/25

Tangle: compact 1-manifold M embedded in $\mathbb{R}\times\mathbb{R}\times[0,1]$ s.t. $\partial M \subset \mathbb{R}\times\{0\}\times\{0,1\}$

M equivalent to M' if they are amb. isotopy rel. to boundary



FRAMED tangles $\rightsquigarrow$ ribbons

RMK: 1. if T has no closed components $\Rightarrow$ T is a braid

2. if T has no "braid components" $\Rightarrow$ T is a link
i.e. $\partial T = \emptyset$

Why tangles?

1. Tangle invariants $\Rightarrow$ braid / link inv.

2. very common:
- category theory
- physics (QFT)
- low-dim. top.

Operator invariants

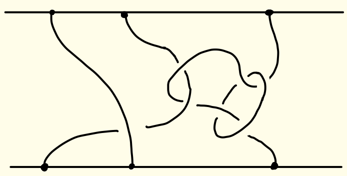
last time: $R: V\otimes V \longrightarrow V\otimes V$ satisfying the YBE on $V^{\otimes 3}$ then we have a representation of B_n

$$B_n \xrightarrow{\psi} \mathrm{End}(V^{\otimes n})$$

We want to do something similar for tangles.

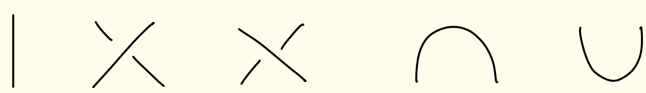
STEP 1: decompose tangles into elementary pieces

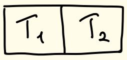
↳ consider a tangle diagram

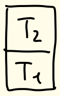


"cut" the diagram to see crossings, caps and cups

THM: every tangle diag. can be obtained from the elementary tangles

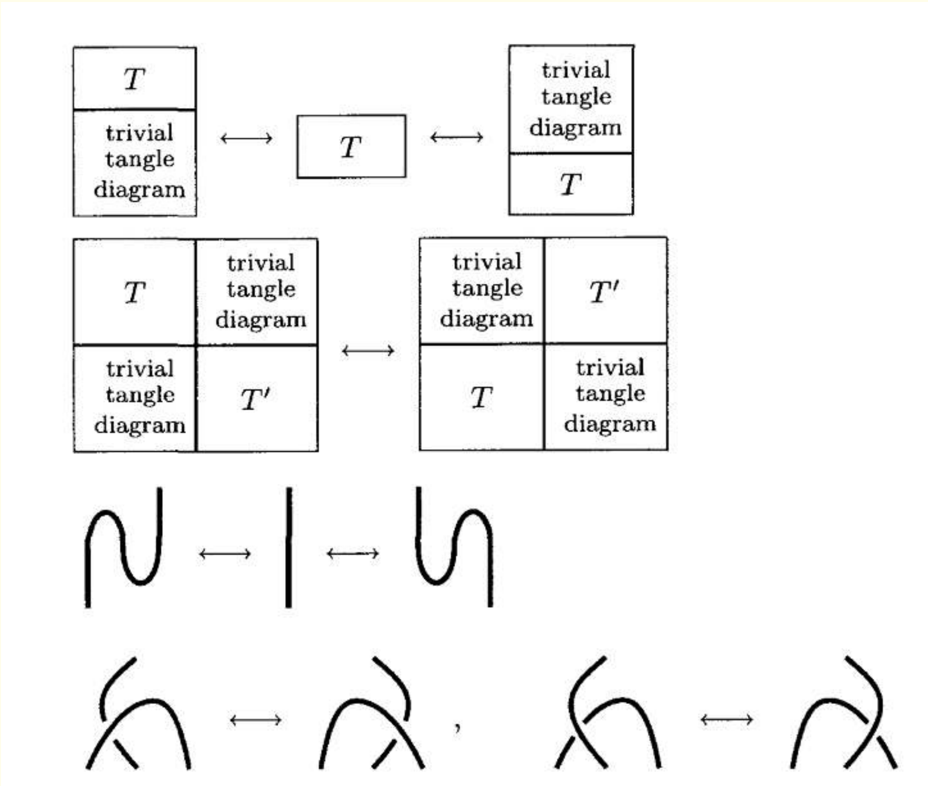


Operations: $T_1 \otimes T_2 =$ 

$T_1 \cdot T_2 =$ 

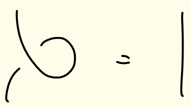
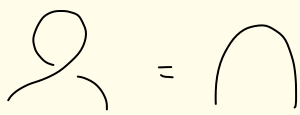
What are the "Markov moves"?

THM (Turaev): two unoriented unframed tangle diagrams describe isotopic tangles iff they are related by a seq. of TURAEV MOVES

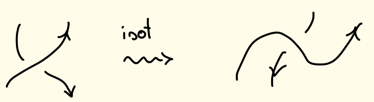


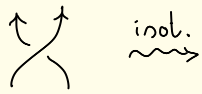
+ Reidemeister moves

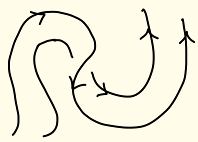
RMK: we can "cut" Reidemeister however we want

ex:  equiv. to 

Oriented version:



; 



crossing have to point down

oriented generators:

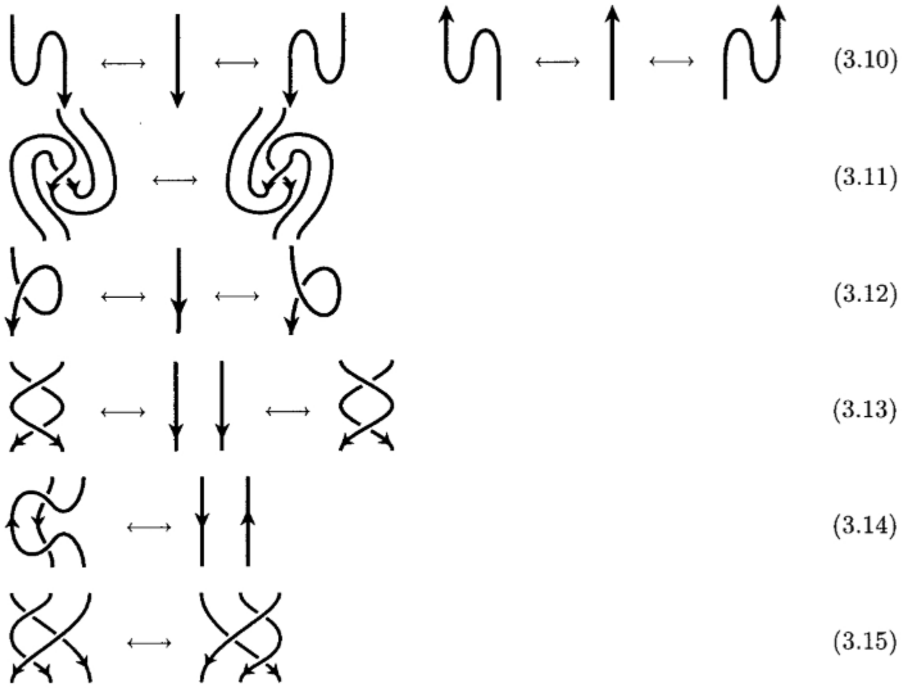
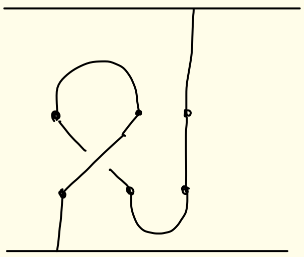


Figure 3.6 The Turaev moves for oriented sliced diagrams

<u>STEP 2</u>: associate to the tangle diagram a linear map



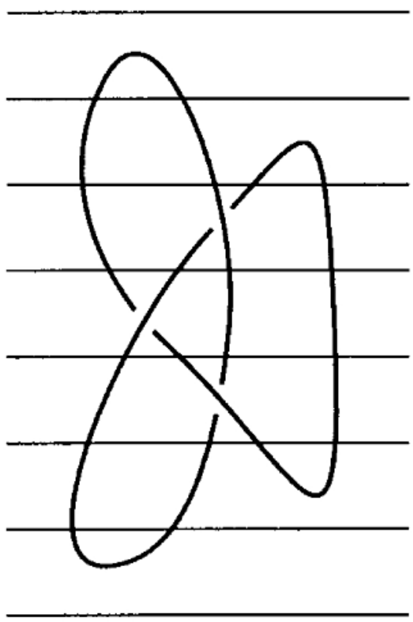
V is a $\mathbb{C}$-vec. space

$u : \mathbb{C} \longrightarrow V \otimes V$

$m : V \otimes V \longrightarrow \mathbb{C}$

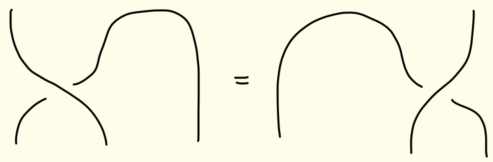
$R : V \otimes V \longrightarrow V \otimes V$

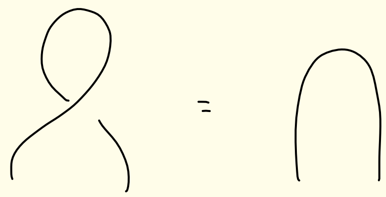
<u>ex</u>:

$T =$ 

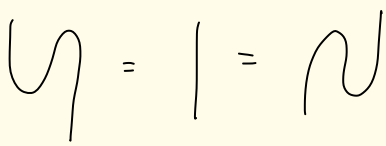
$$\begin{array}{c}
\mathbb{C} \\
\uparrow n \\
V \otimes V \\
\uparrow \mathrm{id}_V \otimes \mathrm{id}_V \otimes n \\
V \otimes V \otimes V \otimes V \\
\uparrow \mathrm{id}_V \otimes R^{-1} \otimes \mathrm{id}_V \\
V \otimes V \otimes V \otimes V \\
\uparrow R \otimes \mathrm{id}_V \otimes \mathrm{id}_V \\
V \otimes V \otimes V \otimes V \\
\uparrow \mathrm{id}_V \otimes R^{-1} \otimes \mathrm{id}_V \\
V \otimes V \otimes V \otimes V \\
\uparrow \mathrm{id}_V \otimes \mathrm{id}_V \otimes u \\
V \otimes V \\
\uparrow u \\
\mathbb{C}
\end{array} = [T]$$

To be an invariant it must respect Turaev moves:

1.  $\Rightarrow (id \otimes m) \circ (R \times id) = (m \otimes id) \circ (id \otimes R)$

2.  $\Rightarrow m \circ R = m$

3. YBE

4.  $\Rightarrow$ m nondeg. lin. form

$(m_{ij})(u^{ij}) = 1$

THM (Turaev): the data $R : V^{\otimes 2} \to V^{\otimes 2}$, $m : V \otimes V \to \mathbb{C}$, $u : \mathbb{C} \to V \otimes V$ satisfying the above make $T \longrightarrow [T]$ a link inv.

TQFT:

$$\text{Tang} = \begin{cases} \text{obj : points in } \mathbb{R} \\ \text{morphisms : cobordisms (tangles) in } \mathbb{R}^3 \end{cases}$$

Braided monoidal functor $\text{Tang} \longrightarrow \text{Vect}_{\mathbb{K}}$