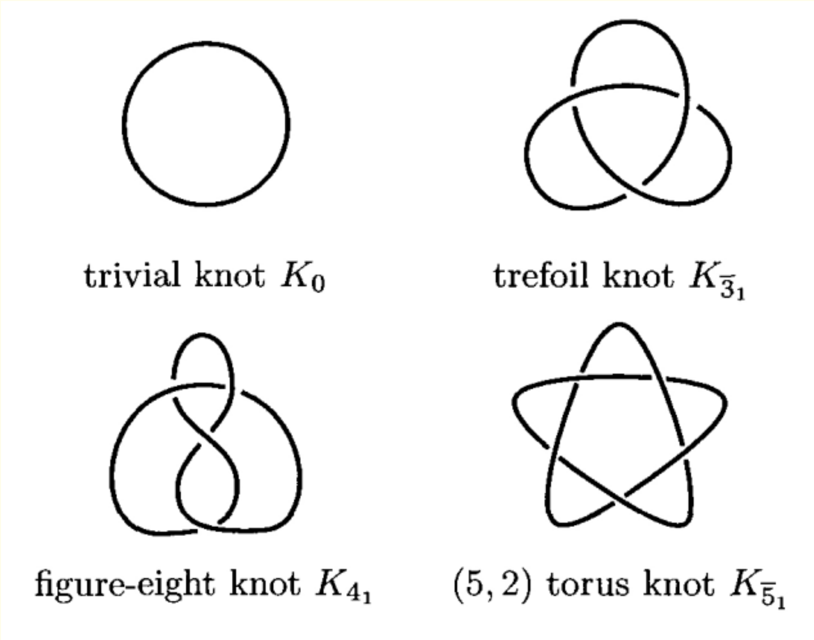
Knots, braids and polynomial invariants

DEF: a KNOT is the image of a smooth embedding $S^1 \longrightarrow \mathbb{R}^3$ (or S^3).

A LINK with m components is the image of $\underbrace{S^1 \sqcup \ldots \sqcup S^1}_{m}$



idea: classify knots and links up to isotopy $\rightsquigarrow$ find isotopy invariants

THM: let L, L' be two links with diagrams Δ, Δ'. Then L is isotopic to L' iff Δ and Δ' are related by a sequence of isotopies of $\mathbb{R}^2$ and Reidemeister moves

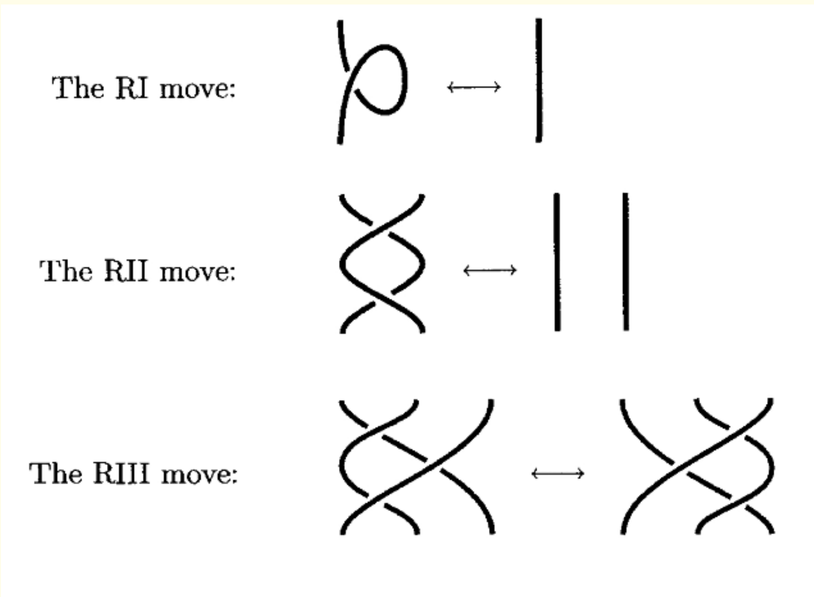


Figure 1.3 The Reidemeister moves

Oriented version :

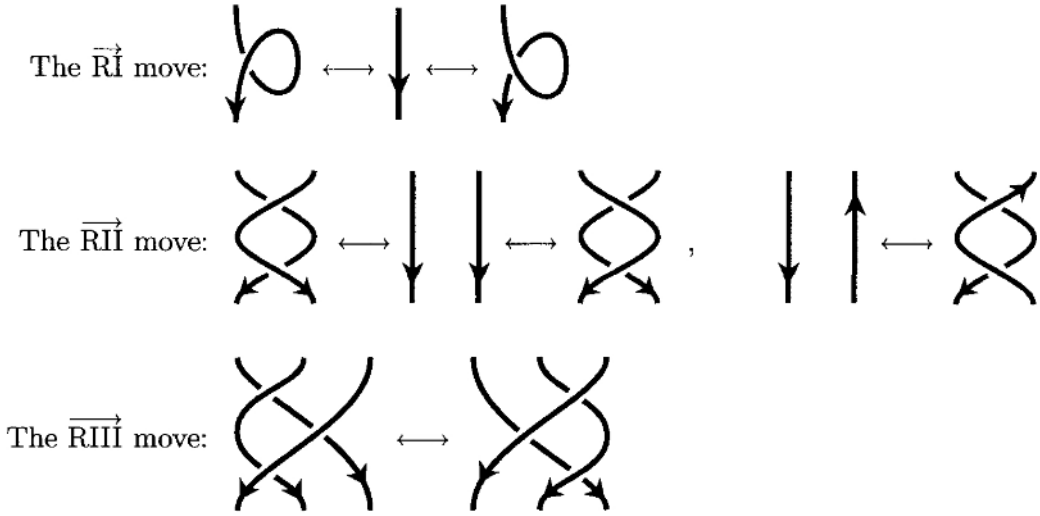


Figure 1.6 The Reidemeister moves for oriented diagrams

Linking number

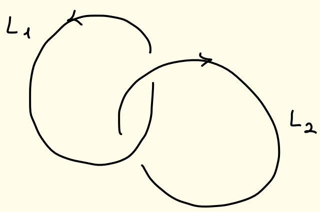
Define positive and negative crossings as follows

positive negative

The LINKING NUMBER of two components L_1 and L_2 of an oriented link with diagrams D_1 and D_2 is :

$$\ell k(L_1, L_2) := \frac{1}{2}\left((\#\text{ pos. crossings of } D_1 \text{ and } D_2) - (\#\text{ neg. crossings of } D_1 \text{ and } D_2) \right)$$

PROP: the linking number is an isotopy invariant of oriented links



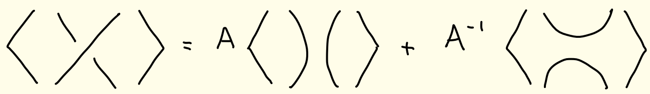
$$\ell k(L_1, L_2) = \frac{1}{2}(2+0) = 1$$

Note : the linking number can also be defined using :

- intersection number of L_2 with Seifert surface of L_1
- homology class of L_2 in $H_1(S^3 \setminus L_1 ; \mathbb{Z})$

JONES POLYNOMIAL

Given a link diagram D in $\mathbb{R}^2$, the ==KAUFFMAN BRACKET== $\langle D\rangle \in \mathbb{Z}[A, A^{-1}]$ is defined recursively as follows:



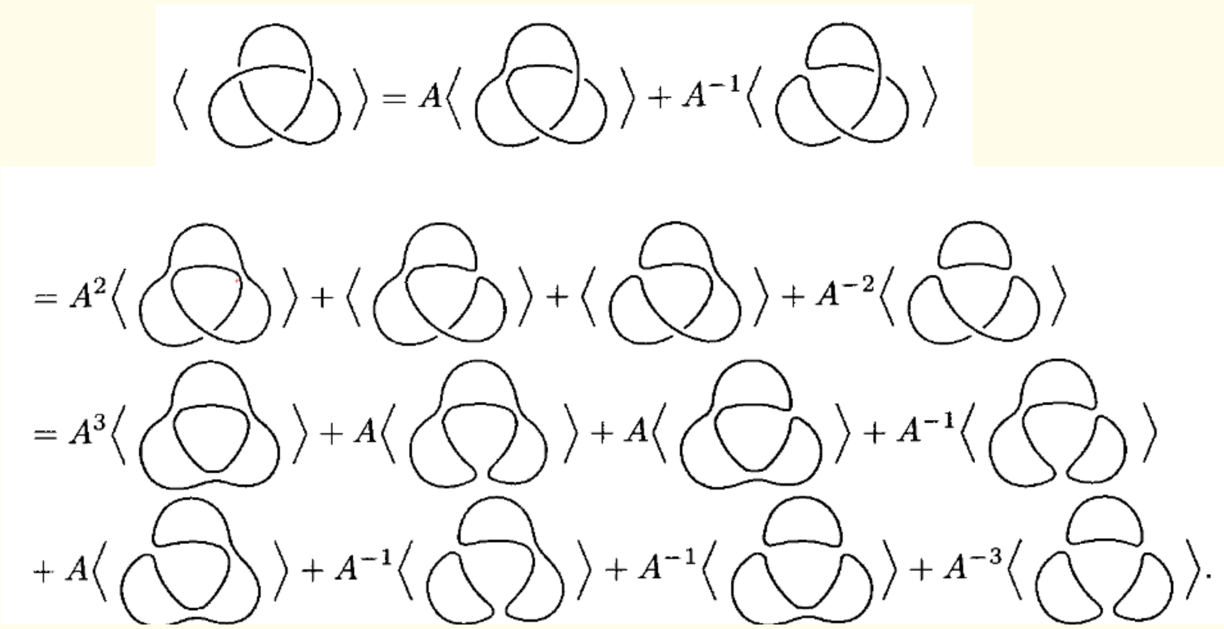
$$\langle \bigcirc \; D \rangle = (-A^2 - A^{-2})\langle D \rangle$$

$$\langle \varnothing \rangle = 1$$

! we will use the convention $\langle \bigcirc \rangle = -A^2 - A^{-2}$ and not $\langle \bigcirc \rangle = 1$!

ex: $\langle \underbrace{\bigcirc \sqcup \ldots \sqcup \bigcirc}_{n \text{ copies}} \rangle = (-A^2 - A^{-2})^n$

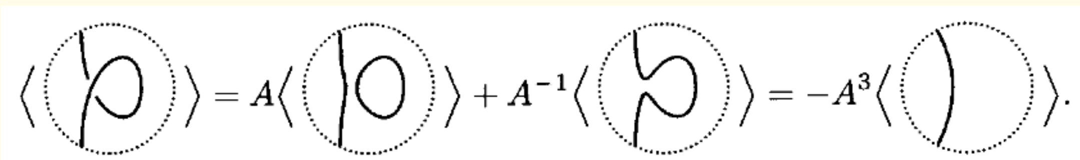
ex:



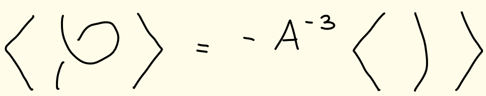
so  $= \underline{\underline{(-A^2 - A^{-2})}}(-A^5 - A^{-3} + A^{-7})$

Invariance under Reidemeister moves:

- R2 ✓ and R3 ✓
- R1:

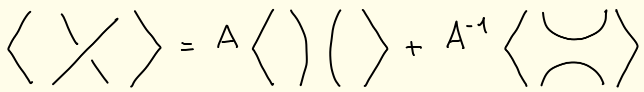


and similarly



We do not have invariance under R1 move.

SKEIN: the Kauffman bracket is characterised by the skein relation



DEF: for an oriented diagram D define the WRITHE of D as:

$$w(D) := (\#\text{ pos. crossings}) - (\#\text{ neg. crossings})$$

THM: L oriented link with diagram D. Then

$$(-A^3)^{-w(D)}\langle D\rangle$$

is an isotopy invariant.

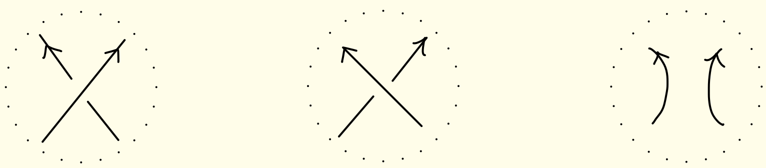
DEF: the JONES POLYNOMIAL of L is

$$V_L(t) := \underline{(-A^2-A^{-2})^{-1}}(-A^3)^{-w(D)}\langle D\rangle\Big|_{A^2=t^{-1/2}} \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$$

It satisfies the following skein relation:

$$t^{-1}V_{L_+}(t) - t\,V_{L_-}(t) = (t^{1/2}-t^{-1/2})\,V_{L_0}(t)$$

where L_+, L_-, L differ locally as follows:



L_+ $\qquad$ L_- $\qquad$ L_0

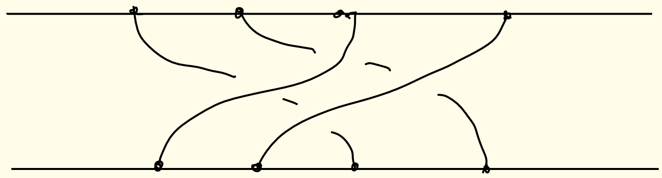
ex: for the trefoil we get

$$V_L(t) = (-A^2-A^{-2})^{-1}(-A^3)^{-3}(-A^5-A^{-3}+A^7)\Big|_{A^2=t^{-1/2}} =$$

$$= t + t^3 - t^4$$

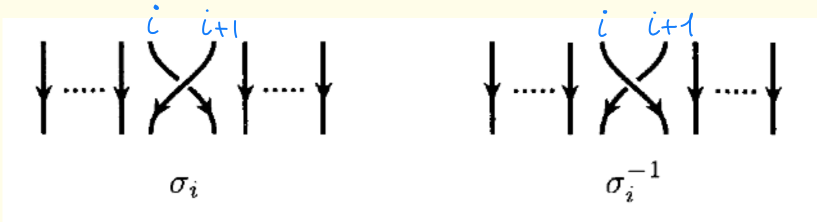
Braids

a BRAID in n strands is a union of n strands embedded in $\mathbb{R}^2 \times [0, 1]$



Denote B_n the braid group in n strands, it has the following presentation

generators: $\sigma_1, \dots, \sigma_{n-1}$



relations: $\sigma_i \sigma_j = \sigma_j \sigma_i \qquad |i-j| \geq 2$

$$\sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i \qquad \forall i = 1 \dots n-2$$

THM (Alexander): any link K is isotopic to the closure of a braid

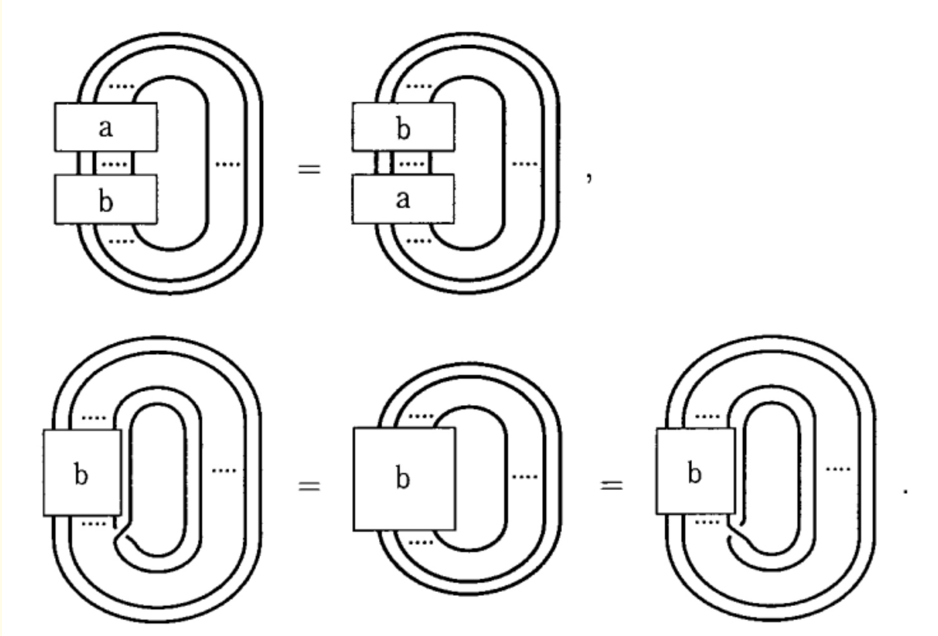
$$\bigcup_{n=1}^{\infty} B_n \longrightarrow \{\text{links}\} / \text{isotopy}$$

Who is the kernel ?

THM (Markov): let b, b' be braids with closure L and L'. Then $L \sim L'$ if and only if b is related to b' by a sequence of Markov moves

$$\text{M I}: \quad ab \longleftrightarrow ba \qquad \forall a, b \in B_n$$

$$\text{M II}: \quad b\sigma_n \longleftrightarrow b \longleftrightarrow b\sigma_n^{-1} \qquad \forall b \in B_n$$



Representation via R matrices

<u>Trace:</u> V $\mathbb{C}$-vec. space, V^* dual

$$V^* \otimes V \cong \mathrm{End}(V) \quad \text{with} \quad f \otimes x \longmapsto (y \longmapsto f(y)\cdot x)$$

$$\text{trace}: \mathrm{End}(V) \xrightarrow{\sim} V^* \otimes V \xrightarrow{ev} \mathbb{C}$$

What if we consider a tensor?

$$\text{trace}_1 : \mathrm{End}(V_1 \otimes V_2) \cong V_2^* \otimes V_1^* \otimes V_1 \otimes V_2 \xrightarrow{\text{contraction}} V_2^* \otimes V_2 \cong \mathrm{End}(V_2)$$

$$\text{trace}_2 : \mathrm{End}(V_1 \otimes V_2) \cong V_2^* \otimes V_1^* \otimes V_1 \otimes V_2 \xrightarrow{\text{contraction}} V_1^* \otimes V_1 \cong \mathrm{End}(V_1)$$

Consider the symmetric group S_m, there is a surjective hom.

$$B_m \twoheadrightarrow S_m$$
$$\sigma_i \longmapsto s_i = (i \ \ i+1$$

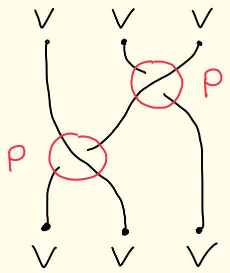
Let $\Psi_m : S_m \longrightarrow \mathrm{End}(V^{\otimes m})$ be the representation given by

$$\Psi_m(s_i) = (id_V)^{\otimes(i-1)} \otimes P \otimes (id_V)^{\otimes(m-i-1)}$$

with $P: x \otimes y \longmapsto y \otimes x$. Then we have a representation

$$\tilde{\Psi}_m : B_m \longrightarrow \mathrm{End}(V^{\otimes m})$$

idea :



What if we change p ?

$$\phi_m : B_m \longrightarrow End(V^{\otimes m})$$
$$\sigma_i \longmapsto (id_V)^{\otimes(i-1)} \otimes R \otimes (id_V)^{\otimes(m-i-1)}$$

to be a rep. we need :

a. $\phi_m(\sigma_i \sigma_j) = \phi(\sigma_j \sigma_i) \quad |i-j| \geq 2$

b. $\phi_m(\sigma_i \sigma_{i+1} \sigma_i) = \phi_m(\sigma_{i+1} \sigma_i \sigma_{i+1}) \quad \forall i = 1 \dots m-2$

a. always satisfied

b. R must satisfy the relation

$$(R \otimes id)(id \otimes R)(R \otimes id) = (id \otimes R)(R \otimes id)(id \otimes R)$$

$$V \otimes V \otimes V \longrightarrow V \otimes V \otimes V$$

so R is a solution of the YANG-BAXTER EQUATION (YBE), called R MATRIX

ex: V 2-dim. , $R \in End(V \otimes V)$

basis $\{e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_0, e_1 \otimes e_1\}$

$$R = \begin{pmatrix} t^{1/2} & & & \\ & 0 & t & \\ & t & t^{1/2} - t^{3/2} & \\ & & & t^{1/2} \end{pmatrix} \in End(V^{\otimes 2})$$

Induces a representation $B_m \xrightarrow{\psi_m} End(V^{\otimes m})$. Consider the trace :

- M I satisfied
- M II ? we need $trace_2 R^{\pm} = 1 \rightsquigarrow$ fails

Modify using $\quad h := \begin{pmatrix} t^{-1/2} & 0 \\ 0 & t^{1/2} \end{pmatrix} \in End(V)$

we get $\mathrm{trace}_2((\mathrm{id}_V \otimes h) R^{\pm}) = \mathrm{id}_V$.

THM: let L be an oriented link and b a braid whose closure is isotopic to L.

Then, for the above Ψ_m and h, we have that

$$\mathrm{trace}\,(h^{\otimes m} \cdot \Psi_m(b))$$

is invariant under Markov moves.

In particular, it is equal to $(t^{1/2} + t^{-1/2})$ times $V_L(t)$.